ON THE SINGULARITY SET OF COMPLEX FUNCTIONS SATISFYING THE CAUCHY-RIEMANN EQUATIONS

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1. INTRODUCTION

Let $f(z) = u(x,y) + iv(x,y)$ be a finite valued complex function defined on a domain D and satisfying the Cauchy-Riemann equations everywhere in D; *i.e.* at every point u and v possess finite first order partials satisfying $u_x = v_y$, $u_y = -v_x$. Under the additional restriction that f be continuous, or even only locally bounded, it is known that f must be analytic in D: theorems of Looman-Menchoff [3], [5] and Tolstov [4] respectively. With no supplementary restriction f need not be analytic everywhere in D (consider *e.g.* $f(z) = e^{-1/z^4}$), but Trokhimchuk ([5, p. 109f]) proved that the singularity set B is a closed totally disconnected set whose projections on the coordinate axes are closed nowhere dense linear sets (a result whose proof required Tolstov's theorem) and asked whether it was possible for B to contain a (perfect) nucleus. It will be shown here that (section 2) B can be non-denumerable and even of positive Lebesgue measure with f satisfying certain additional imposed conditions. Some further questions are raised in section 3.

2. LARGE SINGULARITY SETS

Definition. A complex function $f(z) = u + iv$ has a *directional derivative* $f'(a;z)$ in the direction $a = e^{i\theta}$ at z if $\lim_{h\to 0^+} (f(z + ah) - f(z))/ah$ exists finitely and equals $f'(a;z)$. In particular if $f'(\pm 1;z)$, $f'(\pm i;z)$ all exist and are equal then f is said to satisfy *CR* at z which is equivalent to u, v having first order partials at z obeying the Cauchy-Riemann equations.

LEMMA 1. *Let $\{a_1, a_2, \ldots\}$ be a countable set of directions, $|a_i| = 1$, and let D be the unit disc $|z| \leq 1$. Then there exists a countable isolated subset $A = \{b_1, b_2, \ldots\}$ of D and disjoint open discs N_i centred on b_i, $i \geq 1$, such that if p_i is the orthogonal projection on the tangent L_i to D at a_i then for each $i \geq 1$ the sets $p_i\bar{N}_j \subseteq L_i$ are disjoint for $j \geq i$. Furthermore A can be chosen so that $K = \bar{A}\setminus A$ (which is closed as A is isolated) has planar measure $mK > 0$ and $p_i K \cap p_i\bar{N}_j = \emptyset$ for $1 \leq i \leq j$.*

Proof. Take a closed nowhere dense linear subset K_i of the line segment $p_iD \cap L_i$ such that $m(D \cap p_i^{-1} K_i) > \pi - 2^{-i}$ and let $K = \bigcap_{i=1}^{\infty} p_i^{-1}K_i \cap D$ so that

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K is closed and $mK > 0$. Let $J = \bigcap_{i=1}^{\infty} p_i^{-1}(L_i \setminus K_i) \cap D$ which by Baire's theorem is dense in D, and let $(q_r)_1^\infty$ be a sequence of points in K such that every point of K is the limit of a subsequence.

Suppose $b_1, b_2, \ldots, b_n \in J$ and disjoint open discs N_i centred on b_i, $1 \leq i \leq n$, have been defined and satisfy (i) $p_i \bar{N}_j$ are disjoint for $i \leq j \leq n$ and each i, $1 \leq i < n$; (ii) $K_i \cap p_i \bar{N}_j = \emptyset$, $1 \leq i \leq j \leq n$; (iii) $\bar{N}_i \cap K = \emptyset$, $1 \leq i \leq n$; (iv) $|b_i - q_i| < 2^{-i}$, $1 \leq i \leq n$. We shall define b_{n+1}, N_{n+1} so that conditions (i)-(iv) continue to hold for $n + 1$.

Certainly there exists $b_{n+1} \in J$ and an open disc N_{n+1} centred on b_{n+1} such that $|b_{n+1} - q_{n+1}| < 2^{-(n+1)}$, $\bar{N}_{n+1} \cap K = \emptyset$, $N_{n+1} \cap N_i = \emptyset$ for $1 \leq i \leq n$; also we can ensure, by taking b_{n+1} sufficiently close to q_{n+1} and radius N_{n+1} sufficiently small, that $p_i \bar{N}_{n+1} \cap p_i \bar{N}_j = \emptyset$, $1 \leq i \leq j \leq n$ (note that $p_i(q_{n+1}) \in K_i$ which is closed and disjoint from $p_i \bar{N}_j$) and that $p_i \bar{N}_{n+1} \cap K_i = \emptyset$, $1 \leq i \leq n + 1$ (note that $p_i(b_{n+1}) \in L_i \setminus K_i$ which is open in L_i).

Continuing this inductive construction the set A and discs N_i are constructed and, since $p_i K \subseteq K_i$, do, with K, satisfy the requirements of the lemma.

COROLLARY. *If $z \in K$ and Z_i is the line (not ray) passing through z in the direction a_i then Z_i does not meet $\bar{N}_j$ for $j \geq i$. Hence there exists $\delta > 0$ such that $z + a_i h \notin \bigcup_{j=1}^{\infty} \bar{N}_j$ for $-\delta \leq h \leq +\delta$.*

THEOREM 1. *There is a complex function $f(z) = u + iv$ defined on the plane which satisfies CR everywhere and has the following properties:*

(i) *f has equal directional derivatives in all directions of a countable set $W = \{a_1, a_2, \ldots\}$ of directions at every point;*

(ii) *f has a bounded singularity set B of planar measure $mB > 0$;*

(iii) *at every point u and v possess partial derivatives of all orders and types with respect to x, y and $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$;*

(iv) (cf. Vitushkin [1]) *for every closed contour C disjoint from B, $\int_C f(z)\,dz = 0$.*

Proof. Let W be arbitrary at present and let $A = \{b_1, b_2, \ldots\}$, N_i, $K = \bar{A} \setminus A$ with $mK > 0$ satisfy the hypotheses of Lemma 1. For $k \geq 1$ let $g_k(z)$ be a function analytic everywhere except for an isolated singularity at b_k and such that for z, z' outside N_k and $|z|, |z'| \leq k$, $|g_k(z)| < 2^{-k}$ and $|g_k(z) - g_k(z')| < 2^{-k}|z - z'|$. Define $f(z) = \sum_{k=1}^{\infty} g_k(z)$ so that f will be defined on the plane and will have singularity set $B = K \cup A$.

By the corollary to Lemma 1 $f'(a_i; z)$ exists for $z \in K$, $a_i \in W$ and

$$f'(a_i; z) = \sum_{k=1}^{\infty} g'_k(z).$$

Further if we take $g_k(z) = c_k g(z - b_k)$ for suitable constants c_k where $g(z) = e^{-1/z^4}$, $g(0) = 0$, and if W is chosen with $\pm 1, \pm i \in W$ and all

$$\arg(a_i) \in \bigcup_{m=0}^{3} ((4m-1)\pi/8, (4m+1)\pi/8)$$

then CR holds everywhere and conditions (i) and (ii) of the theorem are satisfied. Provided $c_k \to 0$ sufficiently rapidly (iii) will also hold, by elementary properties of the function $g(z)$.

Finally let C be a closed contour of length l disjoint from B and let c_k have been chosen with $\sum |c_k| < \infty$. Since B and C are compact the distance $d(B,C) = d > 0$. Let M be the upper bound of $|g(z)|$ in the annulus $d \leq z \leq c+1$ where c is the greatest distance of a point of C from 0. Then $|g(z - b_k)| \leq M$ for $z \in C$ and $k \geq 1$, and

$$\sum_{k=1}^{\infty} \int_C |g_k(z)|\, dz \leq Ml \sum |c_k| < \infty.$$

It is therefore justified to interchange the order of summation and integration and deduce,

$$\int_C f(z)\, dz = \int_C \sum_{k=1}^{\infty} g_k(z)\, dz = \sum_{k=1}^{\infty} \int_C g_k(z)\, dz = 0.$$

THEOREM 2. *There is a complex function $f(z)$ defined on the plane which satisfies CR everywhere and has the following properties:*

(i) *f has equal directional derivatives in all directions at every point;*

(ii) *f has a bounded nowhere dense linear singularity set B of positive* 1*-dimensional measure;*

(iii) *for every closed contour C disjoint from B,* $\int_C f(z)\, dz = 0$.

Proof. Let L be the line segment $y = 0$, $0 \leq x \leq 1$ and K any perfect nowhere dense subset of L of positive linear measure. Let b_i, $i \geq 1$, be the midpoints of the disjoint open intervals composing $L \setminus K$ and let N_i be open discs centred on b_i such that N_i subtends an angle less than $1/i$ at the endpoints of the interval containing b_i. Let $A = \{b_1, b_2, \ldots\}$: we shall repeat the construction in the proof of theorem 1 using a function $g(z)$ whose existence is established in the following lemma.

LEMMA 2. *There is a function $g(z)$ analytic everywhere except for an isolated singularity at* 0 *and having the properties:*

(i) *$g(z)$ and $g'(z)$ are bounded in the plane excluding the bounded open region S which is the image of the half-strip T: $x > 1$, $0 < y < 1$ under the (multivalued) mapping $z \to z^{-1/4}$;*

(ii) *if* $g(0)$ *is defined as* 0 *then* $g'(a;0)$ *exists with value* 0 *for all directions* a.

Proof of Lemma. By results [2] on the approximation of analytic functions by entire functions there exists a non-constant entire function $h(z)$ bounded in the plane outside T. For example approximate $\exp(-(z-1)^{1/4})$ defined suitably on the domain T^c. By adding a constant we may suppose $h(0) = 0$.

Define $g(z) = z^2 \int_\infty^z h(z^{-4})\,dz$ where the path of integration is not to pass through 0: since the residue of $h(z^{-4})$ at 0 is 0, $g(z)$ will be defined and single valued for all $z \neq 0$. Conditions (i) and (ii) are then satisfied.

Now set $f(z) = \sum_{k=1}^{\infty} c_k g(z - b_k)$. Provided $c_k \to 0$ sufficiently fast, clauses (i) and (ii) of the theorem will be satisfied with $B = K$. Indeed, for $z \in K \setminus A$ and any direction $a \neq \pm 1$ there is a line segment in direction a containing z in its interior and not meeting $\bigcup_{i=1}^{\infty} \bar{N}_i$, whence, with small enough c_k, $f'(a;z) = \sum_{k=1}^{\infty} c_k g'(z - b_k)$; conditions (i) and (ii) of the lemma then ensure that this equation also holds for $a = \pm 1$ and for $z \in A$ and any a provided a term $g'(0)$ is taken directionally.

Finally clause (iii) follows when $\sum |c_k| < \infty$ by the same argument as used in the proof of clause (iv) in Theorem 1.

3. FURTHER QUESTIONS

The preceding results suggest the following problems:

(a) The constructions of Theorems 1 and 2 depend essentially on the existence of a set of isolated singularities with various properties. Can these be removed, so that in each case f is constructed with a perfect singularity set B?

(b) Does $f(z)$ exist satisfying *CR* everywhere and with equal directional derivatives in all directions at every point, and such that its singularity set is of positive planar measure?

(c) An affirmative answer to (b) would require the existence of a closed totally disconnected set B of planar measure $mB > 0$ such that *every* orthogonal projection pB on a line should be a closed nowhere dense linear set. Can such a set exist?

REFERENCES

1. J. Garnett, *Analytic Capacity and Measure.* Springer Lecture Notes No. 297 (1972), 95f.
2. S. N. Mergelyan, *Uniform approximations of functions of a complex variable.* Uspehi Mat. Nauk (N.S) 7, no. 2 (48) (1952) 31–122. MR 14 (1953), 547.
3. S. Saks, *Theory of the Integral.* Second revised edition. English translation by L. C. Young. With two additional notes by Stefan Banach. Dover Publications, Inc., New York, 1964.

4. G. P. Tolstov, *Sur les fonctions bornées verifiant les conditions de Cauchy-Riemann*. Mat. Sbornik, 52 (1942), 79–85. MR 4 (1943), 136.

5. Ju. Ju. Trohimčuk, *Continuous mappings and conditions of monogeneity*. Israel Program for Scientific Translations, Jerusalem; Daniel Davey & Co., Inc., New York, 1964.

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