TURÁN'S SECOND THEOREM ON SUMS OF POWERS OF COMPLEX NUMBERS

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Let $z_1, \ldots, z_n, b_1, \ldots, b_n$ be complex numbers such that $1 = |z_1| \geq |z_2| \geq \ldots \geq |z_n|$ and define $S_k = b_1 z_1^k + \ldots + b_n z_n^k$. P. Turán [3] considered the problem of finding a lower bound for

$$M_{m,n} = \min \max_{m+1 \leq k \leq m+n} |S_k|,$$

where the min is taken over all possible values of $z_1, \ldots, z_n$ subject to the above constraints. He proved in [3] that

$$M_{m,n} \geq \left(\frac{n}{24e^2(m+2n)} \right)^n \min_{1 \leq j \leq n} |b_1 + \ldots + b_j|$$

and applied this result to various problems, including the question of the distribution of the zeros of $\zeta(s)$ in the critical strip.

Later V. T. Sos and P. Turán [2] improved the estimate by showing that

$$M_{m,n} \geq \left(\frac{n}{A(m+n)} \right)^n \min_{1 \leq j \leq n} |b_1 + \ldots + b_j| \tag{1}$$

holds with $A = 2e^{1+4/e}$. It was pointed out by Uchiyama [4] that the method of [2] will actually give (1) with the better constant $A = 8e$. In fact, it is not hard to see that using the same method one can get

$$M_{m,n} \geq \left(\frac{m}{m+n} \right)^m \left(\frac{n}{8(m+n)} \right)^n \min_{1 \leq j \leq n} |b_1 + \ldots + b_j|;$$

here the factor $(m/(m+n))^m$ always exceeds e^{-n} but tends to e^{-n} as $m \to \infty$.

In this paper we give a further improvement of the constant A in (1); our result is $A \leq 7.81e$. At the cost of some complications, our method could undoubtedly be modified to give a slightly smaller constant.

The problem of finding a lower bound for the best possible constant A in (1) has been considered. The best known result is $A \geq 4e$, due to Makai [1].

We need the following lemma in our proofs.

LEMMA. *Let m be a positive integer and let $z_1, \ldots, z_n$ be any complex numbers.*

Received December 21, 1977.

Michigan Math. J. 26 (1979).

Then there is a δ *with* $m/(m+n) \le \delta \le 1$ *such that for all* z *with* $|z| = \delta$ *the inequality*

$$\left| \prod_{i=1}^{r} (z - z_i) \right| \ge 2 \left(\frac{n}{4(m+n)} \right)^n$$

holds for each $r = 1, 2, \ldots, n$.

Proof. The lemma follows easily from Chebyshev's inequality; see the lemma of Sos and Turán [2, pp. 246–247].

The key fact which we need to obtain our improved estimate for A in (1) is:

THEOREM 1. *Let* m *and* n *be positive integers and let* $x_1, \ldots, x_n$ *be real numbers such that* $1 = x_1 \ge x_2 \ge \ldots \ge x_n \ge 0$. *Define* $f(x) = x^m (x - x_1) \cdot \ldots \cdot (x - x_n)$. *Then*

$$\max_{0 \le x \le 1} |f(x)| \ge \left(\frac{n}{A(m+n)} \right)^n, \tag{2}$$

where $A = 3.905\, e$.

Proof. First suppose that $m \le \mu n$, where $\mu \ge 3$ is a parameter to be chosen later. The function $x^x/(x+1)^x$ decreases if $x \ge 0$, so by the Lemma we have

$$\max_{0 \le x \le 1} |f(x)| \ge 2 \left(\frac{n}{4(m+n)} \right)^n \left(\frac{m}{m+n} \right)^m \ge 2 \left(\frac{n}{4(m+n)} \right)^n \left(\frac{\mu}{\mu+1} \right)^{\mu n} \tag{3}$$

whenever $m \le \mu n$.

Now suppose that $m > \mu n$. Define $H = \alpha n/(m+n)$, where α is a parameter satisfying $1 < \alpha < 2$, and let k be the largest integer less than or equal to n such that the interval $[1 - Hkn^{-1}, 1]$ contains $x_1, \ldots, x_k$. We consider three different cases.

Case 1. $k = n$. We choose

$$x = 1 - \frac{n(m+n+\alpha m)}{(m+n)^2} = \frac{m}{m+n}\left(1 - \frac{\alpha n}{m+n}\right);$$

it follows from the definition of $f(x)$ that

$$\begin{aligned} f(x) &\ge \left(\frac{m}{m+n}\right)^m \left(1 - \frac{\alpha n}{m+n}\right)^m \left(\frac{n(m+n+\alpha m)}{(m+n)^2} - \frac{\alpha n}{m+n}\right)^n \\ &= \left(\frac{n}{m+n}\right)^n \left(\frac{m}{m+n}\right)^m \left(1 - \frac{\alpha n}{m+n}\right)^{m+n} \end{aligned}$$

Calculation shows that the function $(x/(x+1))^x (1 - \alpha(x+1)^{-1})^{x+1}$ increases if $x \ge 3$, so we obtain

(4) $$\max_{0\le x\le 1} |f(x)| \ge \left(\frac{n}{m+n}\right)^n \left(\frac{\mu}{\mu+1}\right)^{\mu n} \left(1-\frac{\alpha}{\mu+1}\right)^{(\mu+1)n}$$

whenever $m > \mu n$.

Case 2. $k \le \beta n$, where $\beta \le 4/(4+e)$ is a parameter.

Let I denote the interval $[1 - Hkn^{-1}, 1]$. Using Chebyshev's inequality, we obtain

$$\max_{x\in I} |(x-x_1)\cdot \ldots \cdot (x-x_k)| \ge 2\left(\frac{Hk}{4n}\right)^k.$$

We also have $x_j < 1 - Hjn^{-1}$ for $k+1 \le j \le n$, so we get

$$\max_{0\le x\le 1} |f(x)| \ge \max_{x\in I} |f(x)| \ge 2\left(1-\frac{Hk}{n}\right)^m \left(\frac{Hk}{4n}\right)^k \cdot \frac{H}{n}\cdot\frac{2H}{n}\cdot \ldots \cdot \frac{(n-k)H}{n}$$
$$> \left(1-\frac{Hk}{n}\right)^m \cdot \frac{(H/e)^n}{4^k}\cdot\frac{(n-k)^{n-k}(ek)^k}{n^n};$$

for the last inequality above we made use of the fact $t! \ge (t/e)^t$.

Calculation shows that the function $(1 - Hxn^{-1})^m (n-x)^{n-x}(ex/4)^x$ decreases if $x \le 4n/(4+e)$, so we obtain

$$\max_{0\le x\le 1} |f(x)| > \left(1-\frac{\alpha\beta n}{m+n}\right)^m \left(\frac{\alpha n}{m+n}\right)^n \cdot 4^{-\beta n}\cdot e^{\beta n - n}\cdot (1-\beta)^{n-\beta n}\cdot \beta^{\beta n}.$$

We also have that

(5) $$\left(1-\frac{a}{x+1}\right)^x \text{ decreases if } x(2-a)+2-2a>0;$$

since $\alpha\beta < 8/(4+e)$ and $m/n > \mu \ge 3$, (5) implies that

$$(1-\alpha\beta n(m+n)^{-1})^m > e^{-\alpha\beta n}.$$

Thus we conclude that

(6) $$\max_{0\le x\le 1} |f(x)| \ge e^{-\alpha\beta n}\left(\frac{\alpha n}{m+n}\right)^n (\beta/4)^{\beta n}((1-\beta)/e)^{n-\beta n}.$$

Case 3. $k > \beta n$. Let γ and δ be two parameters such that $\gamma > \delta > k/n$. Let J denote the interval $[1-\gamma H, 1-\delta H]$. Using Chebyshev's inequality, we obtain

$$\max_{x\in J} |(x-x_{k+1})\cdot \ldots \cdot (x-x_n)| \ge 2(\gamma-\delta)^{n-k}(H/4)^{n-k}$$

so

$$\max_{0\le x\le 1} |f(x)| \ge \max_{x\in J} |f(x)| \ge x_0^m (1 - x_0 - Hk/n)^k (\gamma - \delta)^{n-k} (H/4)^{n-k}$$

where x_0 is some number in J. It is easily verified that for x in J, the function $x^m(1 - x - Hkn^{-1})^k$ takes its minimum at one of the endpoints of J. Thus

$$x_0^m (1 - x_0 - Hk/n)^k \ge \min\left((1-\gamma H)^m (\gamma H - Hk/n)^k, (1-\delta H)^m (\delta H - Hk/n)^k\right)$$

and so we obtain

$$\tag{7} \max_{0\le x\le 1} |f(x)| \ge \left(\frac{\gamma-\delta}{4}\right)^{n-k} \left(\frac{\alpha n}{m+n}\right)^n \times \min\left(\left(1 - \frac{\alpha\gamma n}{m+n}\right)^m \left(\gamma - \frac{k}{n}\right)^k, \left(1 - \frac{\alpha\delta n}{m+n}\right)^m \left(\delta - \frac{k}{n}\right)^k\right).$$

Now we consider the right-hand sides of the inequalities (3), (4) and (6), and we define

$$A_1 = \frac{1}{4}\left(\frac{\mu}{\mu+1}\right)^{\mu}, \quad A_2 = \left(\frac{\mu}{\mu+1}\right)^{\mu}\left(1 - \frac{\alpha}{\mu+1}\right)^{\mu+1},$$
$$A_3 = e^{-\alpha\beta}\alpha(\beta/4)^{\beta}((1-\beta)/e)^{1-\beta}.$$

Also, we define $g(\gamma,\delta,k/n)$ to be the n-th root of, $((m+n)/n)^n$ times the right-hand side of (7). Plainly (2) is proved if we can choose the parameters α, β, μ in such a way that

$$\tag{8} \min(A_1, A_2, A_3, \min_{m>\mu n} \min_{\beta\le k/n\le 1} \max_{\gamma>\delta>k/n} g(\gamma,\delta,k/n)) > A^{-1} = .094207 \ldots.$$

The choice we make is

$$\alpha = 1.05, \quad \beta = .49, \quad \mu = 20.$$

Then calculation gives

$$A_1 = .094222\ldots, \quad A_2 = .128\ldots, \quad A_3 = .095\ldots,$$

so we need only consider the last of the four numbers inside the min in (8). To do this, we let $t = k/n$, $\gamma = yt$, $\delta = zt$ and we define $R(u,v)$ by

$$R(u,v) = \min_{m>\mu n} \min_{u\le t\le v} \max_{y>z>1} \alpha t \left(\frac{y-z}{4}\right)^{1-t} \min(h(y), h(z))$$

where $h(x) = (1 - \alpha x t n(m+n)^{-1})^{m/n}(x-1)^t$. To complete the proof of (8), we need only show that $R(.49, 1) > A^{-1}$. For this we consider four different cases.

Case A. $.49 \le t \le .57$. We choose $y = 3.12$, $z = 1.32$.

It follows from (5) that $h(x) \ge e^{-\alpha x t}(x-1)^t$ for $x = y$ or z, and t in the given interval. Furthermore, as a function of t the expression $t(4(x-1)e^{-\alpha x}(y-z)^{-1})^t$ is increasing for $x = y$ or z, so we may fix $t = .49$ in estimating $R(.49,.57)$. We find that $R(.49,.57) > .099 > A^{-1}$.

Case B. $.57 \le t \le .65$. We choose $y = 2.87$, $z = 1.4$.

It follows from (5) that $h(x) \ge e^{-\alpha x t}(x-1)^t$ for $x = z$ and t in the given interval. We also have

$$h(y) \ge \min(e^{-\alpha y t}, (1 - \alpha y t/21)^{20})(y-1)^t \tag{9}$$

since $m/n > \mu = 20$; calculation shows that $e^{-\alpha y t}$ actually gives the minimum. As in case A, we may fix $t = .57$ in estimating $R(.57,.65)$, and we find that $R(.57,.65) > .099 > A^{-1}$.

Case C. $.65 \le t \le .7$. We choose $y = 2.67$, $z = 1.48$.

Proceeding in the same way as in case B, we obtain the estimate

$$R(.65,.7) > .1 > A^{-1}.$$

Case D. $.7 \le t \le 1$. We choose $y = 2.57$, $z = 1.53$.

As in case B, we have $h(z) \ge e^{-\alpha z t}(z-1)^t$ for t in the given interval and also (9) holds. In this case, neither of the two numbers inside the minimum in (9) is smaller than the other for all values of t such that $.7 \le t \le 1$. Calculation shows that $R(.7,1) > .101 > A^{-1}$. This completes the proof that $R(.49,1) > A^{-1}$, so Theorem 1 is proved.

THEOREM 2. *Let m,n be positive integers; then*

$$M_{m,n} \ge \left(\frac{n}{7.81e(m+n)}\right)^n \min_{1 \le j \le n} |b_1 + \dots + b_j|.$$

Proof. The theorem can be proved using the method of Sos and Turán [2]. Here we give a simpler proof.

Let $z_1, \dots, z_n$ be any complex numbers such that $1 = |z_1| \ge |z_2| \ge \dots \ge |z_n|$. Define

$$f_i(z) = \prod_{j=1}^{i} (z - z_j) \quad (1 \le i \le n), \qquad f_n(z) = f(z),$$

and consider the polynomial

$$P(z) = z^{m+1} \sum_{j=1}^{n} f(z)(z - z_j)^{-1} \cdot \frac{z_j^N}{f'(z_j)\, z_j^{m+1}(z_j^N - \delta^N)}$$

where N is a large positive integer and δ is a number satisfying $0 < \delta < 1$ and

$$|\delta^m(\delta-1)(\delta-|z_2|)\cdot\ldots\cdot(\delta-|z_n|)| \geq \left(\frac{n}{3.905e(m+n)}\right)^n. \tag{10}$$

Such a δ exists by Theorem 1

Define c_i for $m+1 \leq i \leq m+n$ by

$$P(z) = \sum_{i=m+1}^{m+n} c_i z^i.$$

Since $P(z_j) = z_j^N(z_j^N - \delta^N)^{-1}$, we have

$$\sum_{j=1}^{n} b_j z_j^N (z_j^N - \delta^N)^{-1} = \sum_{j=m+1}^{m+n} c_j \left(\sum_{i=1}^{n} b_i z_i^j\right).$$

Letting $N \to \infty$ we get

$$\max_{m+1\leq j\leq m+n} |S_j| \geq \frac{\lim_{N\to\infty}\left|\sum_{j=1}^{n} b_j z_j^N (z_j^N - \delta^N)^{-1}\right|}{\sum_{j=m+1}^{m+n}|c_j|} = \frac{\left|\sum_{j=1}^{K} b_j\right|}{\sum_{j=m+1}^{m+n}|c_j|}, \tag{11}$$

where K is an integer satisfying $1 = |z_1| \geq |z_2| \geq \ldots \geq |z_K| > \delta > |z_{K+1}|$. Thus we need an upper bound for

$$\text{Norm}\,(P(z)) \equiv \sum_{j=m+1}^{m+n} |c_j|.$$

We need the identities

$$(z - z_j)^{-1} = \sum_{k=1}^{j} \frac{f_{k-1}(z_j)}{f_k(z)} \qquad (1 \leq j \leq n) \tag{12}$$

and

$$\frac{f_{k-1}(z)}{f(z)} = \sum_{j=k}^{n} \frac{f_{k-1}(z_j)}{f'(z_j)}(z - z_j)^{-1} \qquad (1 \leq k \leq n). \tag{13}$$

Putting (12) in the definition of $P(z)$ gives

$$P(z) = z^{m+1} \sum_{j=1}^{n} \frac{f(z)\, z_j^N}{f'(z_j)\, z_j^{m+1}(z_j^N - \delta^N)} \sum_{k=1}^{j} \frac{f_{k-1}(z_j)}{f_k(z)}$$

$$= z^{m+1} \sum_{k=1}^{n} \frac{f(z)}{f_k(z)} \sum_{j=k}^{n} \frac{f_{k-1}(z_j)\, z_j^N}{f'(z_j)\, z_j^{m+1}(z_j^N - \delta^N)}.$$

Since $|z_j| \leq 1$, we have trivially that Norm $(z^{m+1} f(z)/f_k(z)) \leq 2^{n-k}$, so

$$\text{(14)} \qquad \operatorname{Norm}(P(z)) \leq \sum_{k=1}^{n} 2^{n-k} \left| \sum_{j=k}^{n} \frac{f_{k-1}(z_j)\, z_j^N}{f'(z_j)\, z_j^{m+1}(z_j^N - \delta^N)} \right|$$

because Norm $(P(z))$ satisfies a triangle inequality. The inner sum (using partial fractions) is

$$\sum_{j=k}^{n} \frac{f_{k-1}(z_j)}{f'(z_j)} \sum_{t=1}^{N} (N\delta^m e^{2\pi i t m/N}(z_j - \delta e^{2\pi i t/N}))^{-1}$$
$$= - \sum_{t=1}^{N} (N\delta^m e^{2\pi i t m/N})^{-1} \sum_{j=k}^{n} \frac{f_{k-1}(z_j)}{f'(z_j)} (\delta e^{2\pi i t/N} - z_j)^{-1}$$
$$= - \sum_{t=1}^{N} (N\delta^m e^{2\pi i t m/N})^{-1} \frac{f_{k-1}(\delta e^{2\pi i t/N})}{f(\delta e^{2\pi i t/N})}$$

by (13). Thus (14) and (10) give

$$\operatorname{Norm}(P(z)) \leq \sum_{k=1}^{n} 2^{n-k} \sum_{t=1}^{N} (N\delta^m)^{-1} \left| \frac{f_{k-1}(\delta e^{2\pi i t/N})}{f(\delta e^{2\pi i t/N})} \right|$$
$$\leq \sum_{k=1}^{n} 2^{n-k} \sum_{t=1}^{N} (N\delta^m)^{-1} |(\delta - |z_1|) \cdot \ldots \cdot (\delta - |z_n|)|^{-1}$$
$$< 2^n \left(\frac{3.905\, e(m+n)}{n} \right)^n.$$

Putting this together with (11) gives Theorem 2.

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