LOCAL EXTENSION OF CR FUNCTIONS FROM WEAKLY PSEUDOCONVEX BOUNDARIES

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Let $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$ be a domain in $\mathbb{C}^n$, $r \in C^2(\mathbb{C}^n)$, $dr \neq 0$ on $\partial\Omega$, and let $\bar\partial_b$ denote the tangential Cauchy-Riemann equations on $\partial\Omega$. A CR-function f on $\partial\Omega$ is a solution of $\bar\partial_b f = 0$; the exact sense in which this equation is interpreted may vary with the regularity of f and $\partial\Omega$. A basic result concerning CR-functions is the following local extension phenomenon, which holds at any strongly pseudoconvex point $p \in \partial\Omega$:

(*) *for each neighborhood* $U' \subset \mathbb{C}^n$ *of* p, *there exists a neighborhood* U'' *of* p *such that each* CR-*function* f *on* $\partial\Omega \cap U'$ *has a holomorphic extension to* $\Omega \cap U''$

(see the references in the survey article by Henkin and Chirka [2]). An important factor in the proof of (*) is that a strongly pseudoconvex boundary can be made (locally) strictly convex by a holomorphic change of coordinates. It is therefore immediate that (*) holds for $f \in \mathcal{O}(\partial\Omega \cap U')$. This local convexity is not true for weakly pseudoconvex domains (see Kohn and Nirenberg [3]), and the proof of (*) in this case is more delicate. Hill and MacKichan [1] have shown that (*) holds for the Kohn-Nirenberg example; they construct a family of disks rather differently from the way it is done below.

THEOREM. *Let* Ω *be a domain in* $\mathbb{C}^n$ *which is real analytic and (weakly) pseudoconvex in a neighborhood of* $p \in \partial\Omega$. *Then* (*) *holds at* p *if and only if there is no germ of a complex variety* V *of codimension one with* $p \in V \subset \partial\Omega$.

Proof. Let us first show that if (*) holds there can exist no germ of a complex hypersurface $V \subset \partial\Omega$. The condition that V has codimension one means that its normal bundle is given by $\partial r \wedge \bar\partial r$ and so it is a manifold. Thus there exists a function f holomorphic in a neighborhood of p such that $\{f = 0\}$ defines V at p and $d\,\mathrm{Re}\, f(p) = dr(p)$. A suitable branch of the function $F(z) = \exp(-f(z)^{-1/2})$ will define a C^∞, CR-function on a neighborhood of p in $\partial\Omega$ which cannot be continued to $\Omega \cap U''$ for any neighborhood U'' of p.

Now we show that (*) holds if V does not exist. More precisely, we will obtain a family of disks satisfying (i) and (ii) below which can be used to construct the extension. (A modern treatment of this is given, for instance, in Polking and Wells [4].) The proof that the function f can actually be extended can be carried

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out just as in the strongly pseudoconvex case. Since we assume that $r(z)$ is real analytic at p, we may introduce a change of coordinates so that $p = 0$ and

$$r(z) = \operatorname{Re}(z_1 + f(z)) + \sum_{\substack{|I| \geq 1 \\ |J| \geq 1}} a_{I,J} z^I \bar{z}^J,$$

where $f(z)$ is holomorphic and vanishes to second order at 0, $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_n)$ are multi-indices, and $z^I = z_1^{i_1}, \ldots, z_n^{i_n}$, $\bar{z}^J = \bar{z}_1^{j_1}, \ldots, \bar{z}_n^{j_n}$. Changing coordinates again by $z_1^* = z_1 + f(z)$, $z_j^* = z_j$, $2 \leq j \leq n$ we may assume that $f = 0$. Now the function $r(0, z_2, \ldots, z_n)$ cannot be identically zero for otherwise $\partial\Omega$ contains a complex hypersurface through 0. Thus in the coordinates $z = (z_1, z')$, we may write

$$r(0, z') = P_k(z') + O(\|z'\|^{k+1})$$

where $P_k = P_k(z_2, \ldots, z_n)$ is a nonzero polynomial homogeneous of degree k.

Now we claim that P_k is a plurisubharmonic function. The defining function for the surface $\partial\Omega$ is given by

$$r(z) = \operatorname{Re} z_1 + P_k(z') + O(|z'|^{k+1}) + O(|z'| \operatorname{Im} z_1) + O(|\operatorname{Im} z_1|^2).$$

Let us compute the Levi form L of $\partial\Omega$. If $(t_1, \ldots, t_n)$ is tangent to $\partial\Omega$, then

$$\sum_{j=1}^{n} t_j \frac{\partial r}{\partial z_j} = 0.$$

Thus at a point $z \in \partial\Omega$ with $\operatorname{Im} z_1 = 0$,

$$t_1 = O(|z'|^{k-1}) |(t_2, \ldots, t_n)|.$$

Since $\partial\Omega$ is pseudoconvex at this point,

$$L(t_1, \ldots, t_n) = O(|z'|^{k-1}) |t|^2 + \sum_{i,j \geq 2} \frac{\partial^2 P_k(z')}{\partial z_i \partial \bar{z}_j} t_i \bar{t}_j \geq 0.$$

The second derivatives of P_k are of order $k - 2$, and so the summation must be non-negative, which shows that P_k is plurisubharmonic.

After a rotation in the $(z_2, \ldots, z_n)$ coordinates, we may assume that

$$P_k(0, \ldots, 0, z_n) \not\equiv 0.$$

Let us set

$$P(z_n) = P_k(0, \ldots, 0, z_n) = \sum_{j=1}^{k-1} a_j z_n^j \bar{z}_n^{k-j}$$

where $a_j = \bar{a}_{k-j}$ and at least one a_j is nonzero. Since P is subharmonic, it follows that $k = 2\ell$ for some $\ell \geq 1$. Subharmonicity (the subaveraging property) also implies that $a_\ell > 0$.

We have

$$P(z_n) = \sum_{0<j<\ell} 2|z_n|^{2j} \operatorname{Re} a_j z_n^{2\ell-2j} + a_\ell |z_n|^{2\ell}.$$

For $\delta > 0$ we define the holomorphic function

$$h(z, \delta) = \sum_{0<j<\ell} 2\,\delta^{2j} a_j z^{2(\ell-j)} + (a_\ell \delta^{2\ell}/2)$$

and the complex manifold $D_\delta = \{(-h(z, \delta), 0, \ldots, 0, z) : |z| < \delta\}$. If $\delta > 0$ is sufficiently small, the following conditions hold:

(i) $D_\delta \cap \Omega \neq \emptyset$

(ii) $\partial D_\delta = \{(-h(z, \delta), 0, \ldots, 0, \delta) : |z| = \delta\}$ is disjoint from $\bar{\Omega}$.

It is clear that (i) holds, since the point $(-(a_\ell \delta^{2\ell}/2), 0, \ldots 0)$ is in the intersection. For (ii), observe that with $h_\delta = h(\delta e^{i\theta}, \delta)$,

$$\begin{aligned} r(-h_\delta, 0, \ldots, 0, \delta e^{i\theta}) &= -\operatorname{Re} h_\delta + O((\operatorname{Im} h_\delta)^2) + O(\operatorname{Im} h_\delta)\, O(\delta) \\ &\qquad + \operatorname{Re} h_\delta + a_\ell \delta^{2\ell}/2 + O(\delta^{2\ell+1}) \\ &= (a_\ell \delta^{2\ell}/2) + O(\delta^{2\ell+1}) \end{aligned}$$

which is positive for δ small.

Remark 1. It would be desirable to remove the hypothesis of real analyticity from the Theorem. The proof given above applies to C^2 boundaries that have the special form $\{0 = \operatorname{Re} z_1 + r(z_2, \ldots, z_n)\}$. It also works when $\partial\Omega$ is C^∞ and does not have the following property:

> *for each integer* $k \geq 1$ *there is a germ of a regular complex hypersurface* M *at* p *such that* $r|_M$ *vanishes to order* k *at* p.

With a different (and easier) proof, the Theorem is true if $\partial\Omega$ is a C^1, convex surface.

Remark 2. If $\partial\Omega \cap U$ is a real analytic subset of U, then

$$S = \{z \in \partial\Omega \cap U : \text{there is a germ } V_z \text{ of an analytic hypersurface } z \in V_z \subseteq \partial\Omega \cap U\}$$

is a closed subset of $\partial\Omega \cap U$.

Proof. Let $H(\partial\Omega)$ denote the holomorphic tangent bundle to $\partial\Omega$. If V_z is an analytic hypersurface, then $TV_z = H(\partial\Omega)|_{V_z}$. Thus for some neighborhood W of z, $V_z \cap W$ may be obtained as the union of all real analytic curves in W starting at z whose tangents lie in $H(\partial\Omega)$. Since $\partial\Omega$ is real analytic, the integrability condition is preserved along each curve, and the union of all curves in U starting at z whose tangents lie in $H(\partial\Omega)$ is a complex submanifold $\tilde{V}_z$, and $V_z \subset \tilde{V}_z \subset \partial\Omega \cap U$. Now if $\{z_j\}$ is a sequence in S converging to z_o, then $\tilde{V}_{z_j}$ converges to $\tilde{V}_{z_o}$. If $\tilde{V}_{z_o}$ contains an open subset of $\partial\Omega$, then $H(\partial\Omega)_z$ is not integrable at some $z \in \tilde{V}_{z_o}$. This contradicts the fact that $H(\partial\Omega)|_{\tilde{V}_{z(j)}}$ is integrable.

We conclude that if $\partial\Omega$ is pseudoconvex and real analytic, then the set where (*) does not hold is closed. If $\partial\Omega$ is pseudoconvex and C^∞, however, this is not true. Let

$$\partial\Omega = \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} w + \varphi(z) = 0\}$$

where φ is convex, $\varphi(0) = 0$, $\varphi \geq 0$, φ is not harmonic in a neighborhood of 0, but there are infinitely many disks clustering at 0 on which φ is linear. It is easily seen that (*) holds at (0,0), but (*) does not hold on the interior of a disk where φ is harmonic.

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