A RESIDUALLY CENTRAL GROUP THAT IS NOT A Z-GROUP

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A group G is *residually central* if $x \notin [x, G]$ holds for every non-identity element x of G. Any Z-group (that is, one which has a central system in the sense of Kurosh [5, p. 218]) is residually central. John Durbin proved in [2] that a locally finite residually central group is a Z-group, and asked whether every residually central group G must be a Z-group (see also Robinson [8, p. 13]). He showed in [3] that this is true if G satisfies the minimum condition for normal subgroups, as did Ayoub in [1]. It is also true and not hard to see that residually central groups G which are either Abelian-by-nilpotent or Abelian-by-locally finite are Z-groups.

It occurred to us that a recent result of P. A. Linnell [6, Theorem A] could be used to give easily understood examples of residually central groups which are not Z-groups. Linnell has shown that if G is a torsion-free polycyclic group with an Abelian subgroup of finite index, then the group algebra KG over any field of nonzero characteristic has no zero divisors.

Suppose G is a nontrivial such group; suppose also that it is residually nilpotent and that G/G' is finite. Let p be a prime not dividing $|G/G'|$ and K the field with p-elements. We form the natural split extension $\Gamma = (KG) \leftthreetimes G$; this, by a well known theorem of P. Hall [4], satisfies the maximum condition on normal subgroups. If $\mathfrak{g}$ denotes the augmentation ideal of KG, then $\mathfrak{g} = \mathfrak{g}^2$ by our choice of p; thus $\mathfrak{g}$ is the limit of the lower central series of Γ. It follows that Γ is not a Z-group.

On the other hand, Γ is residually central. For if x is nonzero in KG, then x cannot lie in $[x, \Gamma]$, since $[x, \Gamma] = x\mathfrak{g}$ and, by Linnell's theorem, an equation $x = x\delta$ cannot hold in KG unless $\delta = 1$. If x is in Γ but not in KG, then (KG)x does not lie in $(KG)[x, \Gamma]$ since G is residually nilpotent. Thus, Γ is residually central.

One example of a group G with all of the stated properties is the group

$$G = \langle x, y \colon x^{-1} y^2 x = y^{-2},\ y^{-1} x^2 y = x^{-2} \rangle$$

discussed by Passman [7, p. 96]. This group has the additional property of being supersoluble.

Thus, there is a finitely generated Abelian-by-supersoluble group Γ *which is residually central and is not a Z-group.*

The second author enjoyed the warm hospitality of Michigan State University during the preparation of this note.

Received June 10, 1977.

Michigan Math. J. 25 (1978).

REFERENCES

1. C. Ayoub, *On properties possessed by solvable and nilpotent groups.* J. Australian Math. Soc. 9 (1969), 218–227.

2. J. Durbin, *Residually central elements in groups.* J. Algebra 9 (1968), 408–413.

3. ———, *On normal factor coverings of groups.* J. Algebra 12 (1969), 191–194.

4. P. Hall, *Finiteness conditions for soluble groups.* Proc. London Math. Soc. (3) 4 (1954), 419–436.

5. A. G. Kurosh, *The theory of groups.* Vol. II. New York, Chelsea, 1956.

6. P. A. Linnell, *Zero divisors and idempotents in group rings.* Math. Proc. Camb. Phil. Soc. 81 (1977), 365–368.

7. D. Passman, *Advances in group rings.* Israel J. Math. 19 (1974), 67–107.

8. D.J.S. Robinson, *Finiteness conditions and generalized soluble groups.* Part 2. Springer-Verlag, New York, 1972.

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