CRITICAL POINTS AND POINT DERIVATIONS ON M(G)

Sadahiro Saeki and Enji Sato

Throughout this paper, let G be an arbitrary *nondiscrete* LCA group, and M(G) the convolution measure algebra of G (*cf.* [8] and [10]). We denote by $\Delta = \Delta_{M(G)}$ the maximal ideal space of M(G). Notice that Δ has a natural semigroup structure; in fact, if S denotes the structure semigroup of M(G), then Δ may be identified with $\hat{S}$, the semigroup of all continuous semicharacters of S [14].

In the present paper we shall study the existence of nontrivial continuous point derivations at certain elements of Δ. Recall that a point derivation at a given element $f \in \Delta$ is a linear functional D on M(G) such that

$$D(\mu * \nu) = (D\mu) \cdot \hat{\nu}(f) + (D\nu) \cdot \hat{\mu}(f), \qquad \mu, \nu \in M(G).$$

We shall say that such a D is continuous if it is continuous in the spectral radius norm of M(G). As is well-known, the existence of a nontrivial continuous point derivation at f implies that f is not a strong boundary point for the uniform closure of $M(G)^\wedge$ in $C(\Delta)$ (see [2; Chapter II, Exercise 12(e)]). On the other hand, the strong boundary points $f \in \Delta$ satisfy $|f|^2 = |f|$ and the Shilov boundary of M(G) is contained in the closure of all such f's ([14; p. 91]). Moreover, if $f \in \Delta$ and $|f|^2 \neq |f|$, then there exists a nontrivial continuous point derivation at f. In fact, letting $f = f_0|f|$ denote the polar decomposition of such an f ([14; p. 28]), we have that $z \to f_0|f|^z$ (Re $z > 0$) is an analytic map having the value f at $z = 1$; hence

$$\mu \to \frac{d}{dz}(\hat{\mu}(f_0|f|^z))\Big|_{z=1}$$

is such a point derivation at f. We may therefore restrict our attention to those elements of Δ which have idempotent modulus. G. Brown and W. Moran [1] have recently proved that there exists a nontrivial continuous point derivation at the critical point of Δ which corresponds to the discrete topology of G. (For a generalization of this result, see [4].) In the present paper we shall prove as a consequence of our main result that the last result holds for every element of Δ whose modulus is a critical point different from the identity $1 \in \Delta$.

Now we introduce some notation. Given a Borel set E in G, let I(E) be the set of those measures μ in M(G) which satisfy $|\mu|(E + x) = 0$ for all $x \in G$, and let $R(E) = I(E)^\perp$ be the set of those measures in M(G) which are singular with respect to all members of I(E). Thus I(E) and R(E) are an L-ideal and an L-subspace of M(G), respectively, and M(G) can be decomposed into the direct sum of I(E)

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and R(E). Moreover, each measure in R(E) is carried by a countable union of translates of E. Let P_E denote the natural projection from M(G) onto R(E). If E is a Borel measurable semigroup in G, then R(E) forms an algebra, and P_E is therefore multiplicative (*cf.* [5]). (By a semigroup in G we mean any subset of G which contains $0 \in G$ and is closed under addition.) In the last case, the linear functional

$$\mu \to (P_E \mu)\hat{}(1) = (P_E \mu)(G)$$

is a complex homomorphism of M(G), which we will denote by h_E.

THEOREM 1. *Let* H *be a* σ*-compact semigroup in* G *such that* $H - H$ *has zero Haar measure, and let* f *be an arbitrary element of* Δ *such that* $|f| \leq h_H$. *Then we have:*

(a) f *is not a strong boundary point for the uniform closure of* $M(G)\hat{}$ *in* $C(\Delta)$;

(b) *If the restriction of* f *to* R(H) *belongs to the Shilov boundary of the algebra* R(H), *then there is a nontrivial continuous point derivation at* f.

Notice that every subgroup of G generated by a σ-compact independent set has zero Haar measure (*cf.* [10]; see also [3], [9] and [12]), and that the condition in part (b) of Theorem 1 is satisfied if $|f| = h_H$. As immediate consequences of Theorem 1, we have the following results.

COROLLARY 1. *If* f *is an element of* Δ *such that* $|f|$ *is a critical point different from* 1, *then there exists a nontrivial continuous point derivation at* f.

COROLLARY 2. *If* f *is a strong boundary point for the uniform closure of* $M(G)\hat{}$ *in* $C(\Delta)$, *then there is no critical point* h *such that* $|f| \leq h \neq 1$.

We shall also prove the following:

THEOREM 2. *Let* H *be a* σ*-compact semigroup in* G *such that* $H - H$ *has zero Haar measure. Then there exists a nontrivial point derivation at* h_H *which is continuous in the total variation norm of* M(G) *but is discontinuous in the spectral radius norm of* M(G).

In order to prove the above results, we need some notation, definitions, and lemmas. For a set K in G, let Gp(K) denote the subgroup of G generated by K. Given a natural number n, we define $K^{(n)}$ to be the set of all sums $x_1 + \dots + x_n$, where the x_j are distinct elements of K, and $nK = K + \dots + K$ (n times). We also define

$$K^{(n)} = nK = 0 \qquad \text{if } n = 0, \quad \text{and}$$

$$nK = (-n)(-K) \qquad \text{if n is a negative integer.}$$

It is easy to show that if K is a σ-compact metrizable subset of G, then all the sets $K^{(n)}$ are σ-compact. Given a subgroup H of G, we shall say that K is *dissociate modulo* H (or H-*dissociate*) if (a) $K \cap H = \emptyset$ and if (b) whenever $x_1, \dots, x_n$ are finitely many distinct elements of K, $(p_1, \dots, p_n) \in \{0, \pm 1, \pm 2\}^n$, and

$$p_1 x_1 + \dots + p_n x_n \in H,$$

then $p_j x_j \in H$ for all $j = 1,2, \dots, n$. Similarly we shall say that K is *independent modulo* H (or H-*independent*) if (a) $K \cap H = \emptyset$ and if (b) whenever $x_1, \dots, x_n$ are finitely many distinct elements of K, $(p_1, \dots, p_n) \in \mathbb{Z}^n$, and

$$p_1 x_1 + \dots + p_n x_n \in H,$$

then $p_j x_j \in H$ for all $j = 1,2, \dots, n$. Notice that when $H = \{0\}$, the above definitions of H-dissociation and H-independence agree with the usual definitions of dissociation and independence, respectively (*cf.* [7] and [10]). Finally we define D(K) to be the union of all $K^{(n)}$ with $n \geq 0$.

LEMMA 1. *Let* H *be a* σ-*compact semigroup in* G, *and let* K *be a Cantor subset of* G *which is dissociate modulo* H_o, *where* $H_o = Gp(H) = H - H$. *Set* $R_o = R(H)$ *and*

$$R_n = R(H + K^{(n)}) \cap I(H + K^{(n-1)}), \qquad n = 1, 2, \dots .$$

Then we have:

(a) *The sets* R_n *are pairwise orthogonal* L-*subspaces of* R(H + D(K));

(b) *Every measure* μ *in* R(H + D(K)) *can be uniquely written as*

$$\mu = \mu_0 + \mu_1 + \mu_2 + \dots ,$$

where $\mu_n \in R_n$ *for* $n = 0,1,2, \dots$ *and* $\|\mu\| = \|\mu_0\| + \|\mu_1\| + \|\mu_2\| + \dots$;

(c) $R_m * R_n \subset R_{m+n}$ *for all* $m,n \in \mathbb{Z}^+$. *In particular,* R(H + D(K)) *forms an* L-*subalgebra of* M(G);

(d) *If* x,y *are two elements of* G *with* $x - y \notin H_o$, *then*

$$|\mu|((H + K^{(n)} + x) \cap (H + K^{(n)} + y)) = 0, \qquad \mu \in R_n, n \in \mathbb{Z}^+.$$

Proof. First we claim that whenever $m,n \in \mathbb{Z}^+$ and $m < n$, then $K^{(m)}$ is covered by finitely many translates of $K^{(n)}$. In fact, this is trivial for $m = 0$. So assume that $m \geq 1$ and that the result is true with m replaced by $m - 1$. Given a natural number n larger than m, take any different $n - m$ elements $x_1, x_2, \dots, x_{n-m}$ of K; then we have

$$K^{(m)} \subset \bigcup_{j=1}^{n-m} \{K^{(m-1)} + x_j\} \cup \{K^{(n)} - (x_1 + \dots + x_{n-m})\}.$$

This, combined with the inductive hypothesis, implies that $K^{(m)}$ is covered by finitely many translates of $K^{(n)}$, and the above claim has been established. It follows at once that

$$\text{(1)} \qquad I(H + K^{(n-1)}) \supset I(H + K^{(n)}), \qquad n = 1, 2, \dots .$$

Part (a) is an easy consequence of (1). Part (b) follows from (a), (1), and the fact that H + D(K) is the union of all $H + K^{(n)}$ with $n \geq 0$.

In order to confirm (c), take any $\mu \in R_m$ and $\nu \in R_n$; we must prove that $\mu * \nu$ is in R_{m+n}. This is trivial if either $\mu * \nu = 0$ or min $(m,n) = 0$, so assume that $\mu * \nu \neq 0$ and $n,m \geq 1$. Since every R_p is a translation invariant L-subspace of M(G), we may also assume that $\mu \in M^+(H + K^{(m)})$ and $\nu \in M^+(H + K^{(n)})$. Under these additional assumptions, it will suffice to prove that $\mu * \nu$ is carried by the set $H + K^{(m+n)}$ and belongs to $I(H + K^{(m+n-1)})$. To this end, take any Borel subset E of G having positive $\mu * \nu$-measure. Then we have

$$\int_G \left[\int_G \xi_E(x + y)\, d\nu(y) \right] d\mu(x) = (\mu * \nu)(E) > 0, \tag{2}$$

where ξ_E denotes the characteristic function of E. Since μ is carried by $H + K^{(m)}$, (2) yields an element $x \in H + K^{(m)}$ such that

$$\int_G \xi_E(x + y)\, d\nu(y) > 0. \tag{3}$$

Let F_1 be any finite subset of K such that $x \in H + F_1^{(m)}$. Since ν is in

$$M(H + K^{(n)}) \cap I(H + K^{(n-1)}),$$

(3) implies that there exists an element y in $(H + K^{(n)}) \setminus (H + K^{(n-1)} + F_1)$ such that $x + y \in E$. Then we have

$$x + y \in \{H + F_1^{(m)}\} + \{H + (K \setminus F_1)^{(n)}\} \subset H + K^{(m+n)}. \tag{4}$$

Thus we have proved that the condition $(\mu * \nu)(E) > 0$ implies $(H + K^{(m+n)}) \cap E \neq \emptyset$, which in turn implies that $\mu * \nu$ is carried by $H + K^{(m+n)}$. To confirm that $\mu * \nu$ is in $I(H + K^{(m+n-1)})$, let p denote the least nonnegative integer such that $H + K^{(p)} + x_o$ has positive $\mu * \nu$-measure for some $x_o = x_o(p) \in G$. Then it is obvious that $p \geq 1$ since ν is in the ideal $I(H + K^{(n-1)}) \subset I(H)$. Moreover, we have $p \leq m + n$ and $(\mu * \nu)((H + K^{(m+n)}) \cap (H + K^{(p)} + x_o)) > 0$ since $\mu * \nu$ is a positive measure carried by $H + K^{(m+n)}$. In particular, we have

$$(H + K^{(m+n)}) \cap (H + K^{(p)} + x_o) \neq \emptyset,$$

so that there is a finite set F_o in K such that $x_o \in H_o + F_o^{(m+n)} - F_o^{(p)}$. Now the minimality of p implies that $(\mu * \nu)(H + K^{(p-1)} + F_o + x_o) = 0$; hence the set $E = (H + K^{(p)} + x_o) \setminus (H + K^{(p-1)} + F_o + x_o)$ satisfies (2). Repeating a similar argument as above, we can therefore find a finite subset F_1 of $K \setminus F_o$ and two elements $x,y \in G$ such that

$$x \in H + F_1^{(m)}, \quad y \in H + (K \setminus (F_o \cup F_1))^{(n)}, \quad \text{and} \quad x + y \in H + (K \setminus F_o)^{(p)} + x_o.$$

Since K is dissociate modulo $H_0 = H - H$ and $p \leq m + n$, the last three conditions imply that $p = m + n$ (and $x_o \in H_o$). It follows from the minimality of p that $\mu * \nu$ is in $I(H + K^{(m+n-1)})$. It is now obvious that $R(H + D(K))$ forms an L-subalgebra of M(G). (In fact, our proof shows that the last result holds without assuming the H_0-dissociation of K.)

Part (d) is essentially proved in [12]. In fact, let x and y be two elements of G with $x - y \notin H_0$, and let $\mu \in R_n$ for some $n \geq 0$. If

$$(H + K^{(n)} + x) \cap (H + K^{(n)} + y) = \emptyset,$$

then there is nothing to prove. So assume that the last intersection is nonempty; then $x - y$ is in $H_o + K^{(n)} - K^{(n)}$ and $n \geq 1$ since $x - y \notin H_o$. Take any finite subset F of K such that $x - y$ is in $H_o + F^{(n)} - F^{(n)}$. Then we have $\{H + (K \setminus F)^{(n)} + x\} \cap \{H + (K \setminus F)^{(n)} + y\} = \emptyset$ by the H_o-dissociation of K, so that $(H + K^{(n)} + x) \cap (H + K^{(n)} + y)$ is contained in the union of

$$H + K^{(n-1)} + F + x \quad \text{and} \quad H + K^{(n-1)} + F + y.$$

Since F is a finite set and μ is in $I(H + K^{(n-1)})$, the last two sets have zero $|\mu|$-measure, which establishes part (d). The proof is complete.

The following lemma is a variant of Lemma 3 of [11].

LEMMA 2. *Let* H *and* K *be as in Lemma* 1, *let* $\mu \in R(H)$, *and let* $\nu_1, \ldots, \nu_n$ *be mutually singular measures in* $M_c(K)$. *If* $(p_1, \ldots, p_n)$ *and* $(q_1, \ldots, q_n)$ *are two different* n*-tuples of nonnegative integers, then we have*

(i) $$\mu * \nu_1^{p_1} * \ldots * \nu_n^{p_n} \perp \mu * \nu_1^{q_1} * \ldots * \nu_n^{q_n},$$

and

(ii) $$\|\mu * \nu_1^{p_1} * \ldots * \nu_n^{p_n}\| = \|\mu\| \cdot \|\nu_1\|^{p_1} \cdot \ldots \cdot \|\nu_n\|^{p_n}.$$

Proof. Replacing H by $H_o = H - H$, we may assume that H is a subgroup of G. Since K is dissociate modulo H, it is obvious that $M_c(K)$ is contained in $R_1 = R(H + K) \cap I(H)$. Setting $p = p_1 + \ldots + p_n$ and $q = q_1 + \ldots + q_n$, we therefore infer from part (c) of Lemma 1 that the measure in the left [right] hand side of (i) is in R_p $[R_q]$. It follows from part (a) of Lemma 1 that the two measures in (i) are mutually singular whenever $p \neq q$. So, assume that $p = q$. If E and F are two different cosets of H, then the measures

(1) $$(\mu|_E) * \nu_1^{p_1} * \ldots * \nu_n^{p_n} \quad \text{and} \quad (\mu|_F) * \nu_1^{q_1} * \ldots * \nu_n^{q_n}$$

are carried by $E + K^{(p)}$ and $F + K^{(p)}$, respectively. It follows from part (d) of Lemma 1 that the measures in (1) are mutually singular. Thus, in order to prove (i), we may assume that μ is carried by a single coset of H, and therefore that μ is carried by H. Now let $K_1, \ldots, K_n$ be any σ-compact disjoint subsets of K such that $\nu_j \in M_c(K_j)$ for $j = 1, \ldots, n$. Then the measures in (i) are carried by

(2) $$H + K_1^{(p_1)} + \ldots + K_n^{(p_n)} \quad \text{and} \quad H + K_1^{(q_1)} + \ldots + K_n^{(q_n)},$$

respectively. Since $(p_1, \ldots, p_n) \neq (q_1, \ldots, q_n)$, the H-dissociation of K assures that the two sets in (2) are disjoint from each other. This establishes (i).

Part (ii) is an easy consequence of (i) (see the proof of Lemma 3 of [11]), and the proof is complete.

LEMMA 3. *Let* H *and* K *be as in the hypotheses of Lemma* 1. *Let* f *be an arbitrary element of* $\triangle$ *such that* $|f| \leq h_H$ *and such that the restriction of* f *to* R(H) *belongs to the Shilov boundary of* R(H). *Then there exists a nontrivial continuous point derivation at* f.

Proof. Let the R_n be as in Lemma 1, and let P_n denote the projection from M(G) onto R_n for $n = 0,1,2, \ldots$. We first prove that

$$\|(P_n\mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty, \qquad \mu \in M(G). \tag{1}$$

If we denote by Q the projection from M(G) onto R(H + D(K)), then Q is multiplicative by part (c) of Lemma 1, so that $\|(Q\mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty$ for all μ in M(G). Moreover, we have $P_n Q = P_n$ for all $n \geq 0$, so it will suffice to establish (1) assuming that μ is in R(H + D(K)).

Given a complex number z of absolute modulus less than or equal to 1 and $g \in \triangle$, define

$$g_z(\mu) = \sum_{n=0}^{\infty} (P_n\mu)^\wedge(g) z^n, \qquad \mu \in R(H + D(K)). \tag{2}$$

Since g is multiplicative, it follows from part (c) of Lemma 1 that g_z is a multiplicative linear functional on R(H + D(K)). Therefore we have

$$|g_z(\mu)| \leq \|\hat{\mu}\|_\infty \qquad \text{for all } \mu \text{ in } R(H + D(K))$$

and all complex numbers z of absolute modulus less than or equal to 1. For a fixed $\mu \in R(H + D(K))$, the right-hand side of (2) is the Fourier expansion of the function $z \to g_z(\mu)$ on the circle group. Hence we have

$$|(P_n\mu)^\wedge(g)| \leq \sup\{|g_z(\mu)| : |z| = 1\} \leq \|\hat{\mu}\|_\infty.$$

Since g is an arbitrary element of $\triangle$, this establishes (1). (The above proof of (1) was suggested by the corresponding proof in [4].)

Next notice that

$$\|[\mu * (\delta_0 + \nu)]^\wedge\|_\infty = \|\hat{\mu}\|_\infty \cdot (1 + \|\nu\|), \qquad \mu \in R(H), \nu \in M_c(K), \tag{3}$$

where δ_0 denotes the unit point measure at $0 \in G$. In fact, we have, by Lemma 2, that

$$\|[\mu * (\delta_0 + \nu)]^n\| = \|\mu^n\| \cdot (1 + \|\nu\|)^n, \qquad n \in \mathbb{Z}^+ \tag{4}$$

for all μ in R(H) and ν in $M_c(K)$. It is evident that (4) implies (3).

Now let $f \in \triangle$ be as in the present lemma. We claim that there is an f_o in $\triangle$ such that

$$\begin{aligned} \hat{\mu}(f_o) &= \hat{\mu}(f) && \text{for } \mu \in R(H), \text{ and} \\ \hat{\nu}(f_o) &= \hat{\nu}(1) && \text{for } \nu \in M_c(K). \end{aligned} \tag{5}$$

To confirm this, let $\triangle_{R(H)}$ denote the maximal ideal space of R(H), and let V be an arbitrary neighborhood of $f|_{R(H)}$ in $\triangle_{R(H)}$. Then the closed set $\triangle_{R(H)} \setminus V$ does not entirely contain the Shilov boundary of R(H). Thus there is a measure $\mu = \mu_V$ in R(H) such that $\|\hat{\mu}\|_\infty = 1$ and $|\hat{\mu}(g)| < 1$ for all g in $\triangle_{R(H)} \setminus V$. Given $\nu_1, \ldots, \nu_n \in M_c^+(K)$, we apply (3) to the above μ and $\nu = \nu_1 + \ldots + \nu_n$; then we can find an element f' in $\triangle$, depending on V and the ν_j, such that

$$\hat{\nu}_j(f') = \hat{\nu}_j(1) \qquad \text{for all } j = 1, 2, \ldots, n$$

and $|\hat{\mu}(f')| = 1$. The last equality and our choice of μ imply that $f'|_{R(H)}$ is in V. Thus a routine weak* argument will yield an $f_o \in \triangle$ satisfying (5).

Finally we define

$$D(\mu) = (P_1 \mu)\hat{}\,(f_o), \qquad \mu \in M(G), \tag{6}$$

where $f_o \in \triangle$ is as in (5). It follows from (1) that D is a linear functional on M(G) which is continuous in the spectral radius norm. To prove that D is a point derivation at f, take any $\mu, \nu \in M(G)$. Recalling that Q is the projection from M(G) onto R(H + D(K)) and that Q is multiplicative, we infer from parts (b) and (c) of Lemma 1 that

$$\begin{aligned} P_1(\mu * \nu) &= (P_1 Q)(\mu * \nu) = P_1[(Q\mu) * (Q\nu)] \\ &= P_1\left[\sum_{m,n=0}^{\infty} (P_m \mu) * (P_n \nu)\right] \\ &= (P_0 \mu) * (P_1 \nu) + (P_0 \nu) * (P_1 \mu). \end{aligned}$$

It follows from (5) and (6) that

$$\begin{aligned} D(\mu * \nu) &= (P_0\mu)\hat{}\,(f_o) \cdot (P_1\nu)\hat{}\,(f_o) + (P_0\nu)\hat{}\,(f_o) \cdot (P_1\mu)\hat{}\,(f_o) \\ &= (P_0\mu)\hat{}\,(f) \cdot D(\nu) + (P_0\nu)\hat{}\,(f) \cdot D(\mu) \\ &= \hat{\mu}(f) \cdot D(\nu) + \hat{\nu}(f) \cdot D(\mu), \end{aligned}$$

where we have used the fact that f vanishes on I(H). Therefore D is a point derivation at f. Finally notice that $M_c(K)$ has a probability measure μ_o since K is a perfect set. Since $M_c(K)$ is contained in R_1, we have

$$D(\mu_o) = \hat{\mu}_o(f_o) = \hat{\mu}_o(1) = 1$$

by (5) and (6), so that D is nonzero. This completes the proof.

LEMMA 4. *Let* H *be a σ-compact subgroup of* G *having zero Haar measure, and let* E(H) *be the set of all integers* p *such that* $p \times U \not\subset H$ *for any neighborhood* U *of* $0 \in G$, *where* $p \times U = \{px : x \in U\}$. *Then there exists a closed metrizable subgroup* G_o *of* G *such that* $p \times V \not\subset H$ *for any* $p \in E(H)$ *and any (relative) neighborhood* V *of* 0 *in* G_o.

Proof. First notice that $1 \in E(H)$, since H has no interior point. By the structure theorem (*cf.* (24.29) of [8] and (2.4.1) of [10]), G contains an open subgroup G_1

of the form $G_1 = \mathbb{R}^N \times J$, where N is a nonnegative integer and J is a compact abelian group. Replacing G and H by G_1 and $G_1 \cap H$, respectively, we may assume that $G = G_1 = \mathbb{R}^N \times J$. If $\mathbb{R}^N \times \{0\}$ is not contained in H, then $G_o = \mathbb{R}^N \times \{0\}$ satisfies the required conclusions. So assume that $\mathbb{R}^N \times \{0\} \subset H$ and write $H = \mathbb{R}^N \times H_1$, where H_1 is the natural projection of H into J. In order to prove the present lemma, we may thus assume that G itself is compact (if necessary, replace G and H by J and H_1, respectively). Assuming this, we let Λ denote the dual group of G. By Theorem (A.15) of [8], Λ can be imbedded in a divisible (discrete) abelian group Γ. By Theorem (A.14) of [8], the dual group K of Γ has the form $K = \Pi\{K_i : i \in I\}$, where each K_i is an infinite, compact, metrizable, abelian group. Denoting by A the annihilator of Λ in K, we see that G is (isomorphic with) K/A. For each subset F of I, we define

$$K(F) = \Pi\{K_i : i \in F\}$$

and regard it as a compact subgroup of K in the usual way. Let π be the quotient map from K onto $G = K/A$. Notice that whenever F is a (at most) countable subset of I, $\pi(K(F))$ is a compact metrizable subgroup of G.

Now choose and fix any p in E(H). We claim that there is a finite or countably infinite subset I_p of I such that whenever V is a neighborhood of 0 in $\pi(K(I_p))$, then $p \times V \not\subset H$. Suppose that there is no finite set having the above property. Write $H = \bigcup_{n=1}^{\infty} H_n$, where (H_n) is an increasing sequence of compact subsets of H. We shall construct a sequence (x_n) of elements of K and a sequence (F_n) of finite subsets of I as follows. Let x_0 be an arbitrary element of $\Pi^*\{K_i : i \in I\}$, the weak direct product of the K_i, and let F_0 be an arbitrary finite subset of I such that $x_0 \in K(F_0)$. Suppose that n is a natural number and that the elements x_k and the sets F_k have been chosen for all $k = 0, 1, \ldots, n-1$ in such a way that $x_k \in K(F_k)$. Put $F'_n = F_0 \cup \ldots \cup F_{n-1}$; then there is an element x_n in $\Pi^*\{K_i : i \in I \setminus F'_n\}$ such that

$$\pi(px_n) \notin H_n - H_n. \tag{1}$$

To see this, take any neighborhood W of 0 in $K(F'_n)$ so that $p \times \pi(W) \subset H$; such a W exists by the present assumption, since F'_n is a finite subset of I. If there is no element x_n as above, then $p \times [\pi(\Pi^*\{K_i : i \in I \setminus F'_n\})]$ must be contained in $H_n - H_n$. Since the last set is compact and since $\Pi^*\{K_i : i \in I \setminus F'_n\}$ is dense in $K(I \setminus F'_n)$, the continuity of π yields $p \times \pi(K(I \setminus F'_n)) \subset H_n - H_n \subset H$. It follows that

$$\begin{aligned} p \times \pi[W \times K(I \setminus F'_n)] &= p \times \{\pi(W) + \pi(K(I \setminus F'_n))\} \\ &= p \times \pi(W) + p \times \pi(K(I \setminus F'_n)) \subset H + H = H. \end{aligned} \tag{2}$$

But π is an open map and $W \times K(I \setminus F'_n)$ is a neighborhood of 0 in K. Thus (2) contradicts our choice of $p \in E(H)$. Now take any finite subset F_n of $I \setminus F'_n$ such that $x_n \in K(F_n)$, which completes the induction.

Setting $I_p = \bigcup \{F_n : n \in \mathbb{Z}^+\}$, we claim that I_p has the required property. Suppose by way of contradiction that there exists a neighborhood V of 0 in $\pi(K(I_p))$ such that $p \times V \subset H$. Then we have

$$V \subset \{x \in \pi(K(I_p)) : px \in H\} = \bigcup_{n=1}^{\infty} \{x \in \pi(K(I_p)) : px \in H_n\}.$$

Since each H_n is compact, it follows from the Baire category theorem that there exists an $n_o \geq 1$ such that $p \times W \subset H_{n_o}$ for some nonempty open subset W of $\pi(K(I_p))$. On the other hand, we have $\{x_n\} \subset K(I_p)$ and $x_n \to 0$ as $n \to \infty$ by the construction. Since π is a continuous homomorphism, it follows that

$$\pi(px_n) = p\,\pi(x_n)$$

belongs to $p \times (W - W) \subset H_{n_o} - H_{n_o}$ for all n large enough. This contradicts (1) and the present claim has been established.

To complete the proof, we define $I' = \bigcup \{I_p : p \in E(H)\}$. It is easy to show that the group $\pi(K(I'))$ has all the required properties. The proof is complete.

Now let q(G) denote the largest member q of $\{2, 3, \ldots, \infty\}$ such that every neighborhood of $0 \in G$ contains an element of order q.

LEMMA 5. *Let* H *be a* σ*-compact subgroup of* G *having zero Haar measure. Then there exists an* H*-independent Cantor set* K *in* G. *If* H *has the property that* $p \times U \not\subset H$ *for any natural number* p *less than* q(G) *and any neighborhood* U *of* $0 \in G$, *then such a* K *can be chosen so that* $\mathrm{Gp}(K) \cap H = \{0\}$.

Proof. We prove this by modifying the well-known method of constructing independent Cantor sets (see [6] or 5.2.4 of [10]). Let G_o be the subgroup of G as in Lemma 3.

There are two possibilities; either (a) there is a natural number q such that $q \times U_q \subset H$ for some neighborhood U_q of 0 in G_o, or (b) there is no natural number q as in (a). In case (a), we define q_o as the least natural number satisfying the condition in (a), G_1 as the corresponding neighborhood U_{q_o} of 0 in G_o, and F_n as the set of all nonzero elements of $\{0, 1, \ldots, q_o - 1\}^{2(n)}$ for $n \geq 1$, where $2(n) = 2^n$. In case (b), we define G_1 as G_o and F_n as the set of all nonzero elements of $\{0, \pm 1, \ldots, \pm n\}^{2(n)}$ for $n \geq 1$. Notice that if (a) is the case, then $q_o \geq 2$, and that if in addition H has the property stated in the last assertion of the present lemma, then $q_o = q(G)$. Let (H_n) be an increasing sequence of compact sets with $H = \bigcup H_n$.

By induction on $n = 0, 1, \ldots$, we shall construct nonempty open subsets $V_n(j)$, $1 \leq j \leq 2^n$, of G_1, as follows. Put $V_0(1) = G_1$, and assume that the sets $V_{n-1}(j)$,

$1 \le j \le 2^{n-1}$, have been defined for some $n \ge 1$. Applying the Baire category argument, we can easily find distinct elements $x_n(2j-1)$, $x_n(2j)$ of $V_{n-1}(j)$ so that

$$\sum_{k=1}^{2(n)} sp_k x_n(k) \notin H, \qquad (p_1, \ldots, p_{2(n)}) \in F_n.$$

Since H_n is a compact subset of H and since F_n is a finite set, there exist neighborhoods $V_n(k)$ of $x_n(k)$ in G_1 such that

$$\left[\sum_{k=1}^{2(n)} p_k V_n(k)\right] \cap H_n = \emptyset, \qquad (p_1, \ldots, p_{2(n)}) \in F_n. \tag{1}$$

Without loss of generality, we may assume that $V_n(2j-1)$ and $V_n(2j)$ have disjoint compact closures contained in $V_{n-1}(j)$ for all $j = 1, 2, \ldots, 2^{n-1}$, and that the diameter of every $V_n(k)$ is less than $1/n$. This completes the induction. We define

$$K = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2(n)} V_n(k), \tag{2}$$

and claim that K has the required property.

In fact, it is easy to show that K is a Cantor set (notice that the definition of K is unchanged even if the sets $V_n(k)$ are replaced by their closures). Suppose that $a_1, \ldots, a_N \in \mathbb{Z}$, that $x_1, \ldots, x_N$ are distinct elements of K, and that

$$a_1 x_1 + \ldots + a_N x_N$$

is in H. We must prove that $a_1 = \ldots = a_N = 0$ in case (b) and that every a_j is a multiple of q_o in case (a). When we deal with case (a), we may and do assume that $0 \le a_j < q_o$ for all $j = 1, \ldots, N$, since $K \subset G_1$ and $q_o \times G_1 \subset H$. Now take any natural number N(1) such that

$$a_1 x_1 + \ldots + a_N x_N \in H_{N(1)}. \tag{3}$$

By our construction of K, there is a natural number N(2) such that whenever $n > N(2)$, the elements x_j belong to different sets in $\{V_n(k): 1 \le k \le 2^n\}$. Choose and fix any natural number n larger than all of the N(1), N(2), $|a_1|, \ldots, |a_N|$. Then (1) and (3) imply $a_1 = \ldots = a_N = 0$, since we have $H_{N(1)} \subset H_n$. Evidently this completes the proof.

LEMMA 6. *Let* H *be a* σ*-compact subgroup of* G *having zero Haar measure, let* G_o *be a metrizable closed subgroup of* G *such that* $H \cap G_o$ *is nonopen in* G_o, *and let* N *be a natural number. Then there exists a Cantor set* K *in* $G_o \setminus H$ *and a probability measure* ρ *in* $M_c(K)$ *having the following properties:*

(a) ρ^{N+1} *is absolutely continuous with respect to* λ_o, *the Haar measure on* G_o;

(b) $(H \cap G_o) + NK$ *has zero* λ_o*-measure;*

(c) *If* $x_1, x_2, \ldots, x_N$ *are different* N *elements of* K, *if* $p_1, p_2, \ldots, p_N \in \mathbb{Z}$, *and if* $p_1 x_1 + p_2 x_2 + \ldots + p_N x_N \in H$, *then* $p_j x_j \in H$ *for all* $j = 1, 2, \ldots, N$.

Proof. Since this is a variant of Theorem 2.4 in [13] (see also Remark 8.3 in [13]), we shall only give a sketch of the proof.

Replacing H by $H \cap G_o$, we may assume that H is contained in G_o. Then notice that H has zero λ_o-measure since H is nonopen in G_o. Let E(H) be the set of all integers p such that $p \times V \not\subset H$ for any neighborhood V of 0 in G_o. We denote by Γ_o and $M_a(G_o)$ the dual group of G_o and the set of those measures in $M(G_o)$ which are absolutely continuous with respect to λ_o, respectively. Thus $M_a(G_o)$ is isometrically isomorphic with the group algebra $L^1(G_o, \lambda_o)$. The Cantor set K and the measure ρ having the required properties will be constructed in three steps.

Step 1. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are measures in $M_a^+(G_o)$ that $D \subset H$ and $Y \subset \Gamma_o$ are compact sets, that $\varepsilon > 0$, and that F is a finite set consisting of elements $(p_1, p_2, \ldots, p_N) \in \mathbb{Z}^N$ such that $p_j \in E(H)$ for at least one index j. Then there exist measures $\mu_1, \mu_2, \ldots, \mu_N$ in $M_a^+(G_o)$, with pairwise disjoint compact supports, such that for each $j = 1, 2, \ldots, N$, we have

(i) $$\|\mu_j\| = \|\lambda_j\|;$$

(ii) $$\operatorname{supp} \mu_j \subset \operatorname{supp} \lambda_j;$$

(iii) $$\lambda_o\left[D + N\left(\bigcup_{k=1}^{N} \operatorname{supp} \mu_k\right)\right] < \varepsilon;$$

(iv) $$|\hat{\mu}_j(\gamma) - \hat{\lambda}_j(\gamma)| < \varepsilon \qquad \text{for all } \gamma \in Y;$$

(v) If $(p_1, p_2, \ldots, p_N) \in F$ and $x_k \in \operatorname{supp} \mu_k$ for all k, then
$$p_1 x_1 + p_2 x_2 + \ldots + p_N x_N \notin D.$$

The existence of the measures $\mu_j \in M_a^+(G_o)$ satisfying (i)-(iv) is a consequence of Lemma 6.1 of [13]. In order to let the μ_j further satisfy (v), it will suffice to apply a routine category argument. We omit the details.

Step 2. Suppose that λ is a measure in $M_a^+(G_o)$, that $D \subset H$ and $Y \subset \Gamma_o$ are compact sets, and that $\varepsilon > 0$. Then there exist finitely many, pairwise disjoint, compact subsets $K_1, K_2, \ldots, K_T$ of supp λ, where $T > \max(N, 1/\varepsilon)$, and a measure μ in $M_a^+(G_o)$ such that

(i) $$\|\mu\| = \|\lambda\| \quad \text{and} \quad \|\mu^{N+1} - \lambda^{N+1}\| < \varepsilon;$$

(ii) $$\operatorname{supp} \mu = \bigcup_{j=1}^{T} K_j \quad \text{and} \quad \operatorname{diam}(K_j) < \varepsilon \qquad \text{for all indices } j;$$

(iii) $$\lambda_o[D + N(\operatorname{supp} \mu)] < \varepsilon;$$

(iv) $$|\hat{\mu}(\gamma) - \hat{\lambda}(\gamma)| < \varepsilon \qquad \text{for all } \gamma \in Y;$$

(v) If $(p_1, p_2, \ldots, p_T) \in \{0, \pm 1, \ldots, \pm T\}^N$ satisfies $p_j \neq 0$ for at most N indices j and $p_j \in E(H)$ for at least one index j, and if $x_j \in K_j$ for all indices j, then

$$p_1x_1 + p_2x_2 + \dots + p_Tx_T \notin D.$$

This can be proved along the same lines as Lemma 6.2 of [13] by applying the result stated in Step 1 and Lemma 3.1 of [13]. We leave the details to the reader.

Step 3. By induction on $n = 0, 1, 2, \dots$, we shall construct a sequence (ρ_n) of probability measures in $M_a(G_o)$ as follows. Let (D_n) be an increasing sequence of compact sets such that $H = \bigcup_{n=1}^{\infty} D_n$, and let ρ_o be any probability measure in $M_a(G_o)$ with compact support disjoint from H. In the case that there is a natural number p such that $p \times V \subset H$ for some neighborhood V of 0 in G_o, we shall also demand that $q_o \times (\text{supp}\,\rho_o) \subset H$, where q_o denotes the least one of all natural numbers p as above. Suppose that n is a natural number and that the probability measures $\rho_j \in M_a^+(G_o)$ have been constructed for all $j = 0, 1, \dots, n-1$. Define

$$Y_n = \{\gamma \in \Gamma_o : |\hat{\rho}_j(\gamma)| \geq n^{-1} \quad \text{for some } j = 0, 1, \dots, n-1\},$$

and notice that Y_n is a compact subset of Γ_o. Setting $\lambda - \rho_{n-1}, D = D_n, Y = Y_n$ and $\varepsilon = 2^{-n}$, we now apply the result in Step 2 to find pairwise disjoint compact subsets K_{nj} $(1 \leq j \leq T_n)$ of supp ρ_{n-1}, where $T_n > N2^n$, and a measure

$$\mu = \rho_n \in M_a^+(G_o),$$

subject to the five conditions given in Step 2. This completes the induction.

It is easy to show that the sequence (ρ_n) converges weak* to a probability measure ρ in $M(G_o)$, and that $K = \text{supp}\,\rho$ and ρ satisfy the required conditions. The proof is complete.

Proof of Theorem 1. Let H be any semigroup in G as in the hypotheses of Theorem 1 and let f be any element of $\triangle$ such that $|f| \leq h_H$. By virtue of Lemma 5, there exists a Cantor set K in G which is independent modulo $H_o = H - H$. It is evident that every H_o-independent set is H_o-dissociate. Thus part (b) of Theorem 1 follows from Lemma 3.

In order to prove part (a), assume that f is a strong boundary point for the uniform closure of $M(G)^{\wedge}$ in $C(\triangle)$. Let V be an arbitrary neighborhood of $f|_{R(H)}$ in $\triangle_{R(H)}$, the maximal ideal space of R(H). Then it is evident that the set of all $g \in \triangle$ with $g|_{R(H)} \in V$ is a neighborhood of f in $\triangle$. By the present assumption, we can therefore find a measure λ in M(G) such that $|\hat{\lambda}(f)| \geq 1$ and $|\hat{\lambda}(g)| < 1$ for all $g \in \triangle$ with $g|_{R(H)} \notin V$. Now write $\lambda = \mu + \nu$, where $\mu \in R(H)$ and $\nu \in I(H)$. Then we have $|\hat{\mu}(f)| = |\hat{\lambda}(f)| \geq 1$ since f vanishes on I(H). On the other hand, every element g of $\triangle_{R(H)}$ extends (uniquely to an element g' of $\triangle$ such that $|g'| \leq h_H$. Therefore we have $|\hat{\mu}(g)| = |\hat{\lambda}(g')| < 1$ for all g in $\triangle_{R(H)} \setminus V$. Since μ is in R(H) and since V is an arbitrary neighborhood of $f|_{R(H)}$ in $\triangle_{R(H)}$, it follows that $f|_{R(H)}$ belongs to the Shilov boundary of R(H). But then part (b) of the present theorem implies that there exists a nontrivial continuous point derivation at f. This yields the required contradiction, since there is no such point derivation at any strong boundary point. The proof is complete.

Proofs of Corollaries 1 *and* 2. Suppose that h is a critical point in Δ different from 1. Let τ be the locally compact group topology for G which naturally corresponds to h (see Chapters 7 and 8 of [14]). Thus τ is strictly stronger than the original topology of G. Let H be any subgroup of G which is open and σ-compact in the topology τ. Then H is a σ-compact subgroup of G which is of the first Baire category in the original topology of G, and we have $h = h_H$. Therefore Corollaries 1 and 2 follow from parts (b) and (a) of Theorem 1, respectively.

Proof of Theorem 2. Suppose that H is a σ-compact semigroup in G such that $H_o = H - H$ has zero Haar measure. By Lemma 4, there exists a metrizable closed subgroup G_o of G such that $H_o \cap G_o$ is nonopen in G_o. Choose and fix any natural number $N \geq 6$, and take any Cantor set K in $G_o \setminus H_o$ and any probability measure $\rho \in M_c(K)$ satisfying the conclusions of Lemma 6 (with H_o in place of H). Let the R_n $(n \geq 0)$ be the L-subspaces of $R(H + D(K))$ defined as in Lemma 1. Notice that $R(H + D(K))$ forms an L-subalgebra of M(G), as was observed in the proof of Lemma 1. Furthermore, part (c) of Lemma 6 with H_o in place of H guarantees that the inclusion $R_m * R_n \subset R_{m+n}$ obtains whenever m,n are non-negative integers satisfying $4(m + n) \leq N + 2$. This can be easily seen from the proof of part (c) of Lemma 1.

Now let P_n denote the natural projection from M(G) onto R_n $(n \geq 0)$, and define

$$D(\mu) = (P_1\mu)\hat{}\,(1) = (P_1\mu)(G), \qquad \mu \in M(G).$$

Since $R_m * R_n \subset I(H + K)$ whenever $m,n \in \mathbb{Z}^+$ and $m + n \geq 2$, the proof of Lemma 3, combined with the above remarks, shows that D is a nontrivial point derivation at h_H. It is evident that D is continuous in the total variation norm of M(G).

In order to confirm that D is discontinuous in the spectral radius norm of M(G), notice that $K^{(N+1)}$ has positive λ_0-measure and that $\hat{\rho}$ belongs to $C_o(\Gamma_o)$ by part (a) of Lemma 6, where Γ_o denotes the dual group of G_o. Therefore we have $M_a(G_o) \subset R(H + K^{(N+1)})$ and, given $\varepsilon > 0$, there is a measure λ_ε in $M_a(G_o)$ such that $\sup\{|\hat{\rho}(\gamma) - \hat{\lambda}_\varepsilon(\gamma)| : \gamma \in \Gamma_o\}$ is less than ε. Since ρ^{N+1} is in $M_a(G_o)$, it follows that $\sup\{|\hat{\rho}(f) - \hat{\lambda}_\varepsilon(f)| : f \in \Delta\} < \varepsilon$. On the other hand, $H \cap G_o + NK$ has zero λ_0-measure by part (b) of Lemma 6, so that $M_a(G_o)$ is contained in $I(H + NK)$; hence, in particular, $M_a(G_o) \perp R_1$. Thus we have

$$D(\rho - \lambda_\varepsilon) = D(\rho) = 1 \quad \text{and} \quad \|\hat{\rho} - \hat{\lambda}_\varepsilon\|_\infty < \varepsilon,$$

so that D is discontinuous in the spectral radius norm of M(G). This establishes Theorem 2.

Remarks. (a) The H_0-dissociation of K in Lemmas 1, 2, and 3 may be replaced by the following weaker condition: if $x_1, x_2, \ldots, x_n$ are finitely many different elements of K and if $(p_1, p_2, \ldots, p_n) \in \{+1, -1\}^n$, then

$$p_1x_1 + p_2x_2 + \ldots + p_nx_n \notin H_o.$$

(b) If, in Lemmas 1, 2, and 3, H forms a subgroup of G and K is independent modulo H, then we can replace the sets $H + D(K)$ and $K^{(n)}$ by $H + \left(\bigcup_{p=0}^{\infty} pK\right)$ and nK, respectively.

(c) Given $f \in \triangle$, write $J_f = \{\mu \in M(G) : \hat{\mu}(f) = 0\}$. If there is a nontrivial continuous point derivation D at f, then D extends to a bounded linear functional on the uniform closure of $M(G)^\wedge$ in $C(\triangle)$. It is evident that f and D are linearly independent as functionals and that D vanishes on the linear span of $J_f * J_f$. It follows that $(J_f)^\wedge$ can not be contained in the closed linear span of $(J_f * J_f)^\wedge$ in $C(\triangle)$. Conversely, if J_f has the last property, then there exists a nontrivial continuous point derivation at f, as was observed by Brown and Moran [1].

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Department of Mathematics
Tokyo Metropolitan University,
Fukazawa, Setagaya, Tokyo 158,
JAPAN

and

Department of Mathematics,
Yamagata University,
Yamagata 990,
JAPAN