APPROXIMATING DIRECT INTEGRALS OF OPERATORS BY DIRECT SUMS

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1. INTRODUCTION

Although most operator theorists feel comfortable with direct sums of operators, many feel uneasy when confronted with direct integrals of operators. Direct integrals are natural analogues of multiplications by complex L^∞-functions on complex L^2-spaces; they are multiplications by operator-valued L^∞-functions on vector-valued L^2-spaces. One major problem is that complex L^∞-functions can be uniformly approximated by simple functions, but operator-valued L^∞-functions cannot even be approximated by functions with countable range. An operator-valued L^∞-function with countable range is really a direct sum (see Lemma D); therefore direct integrals cannot be "naturally" approximated by direct sums. It is the purpose of this note to show that, if one is willing to leave the measure-theoretic structure, direct integrals can be approximated by direct sums (Theorem A). As a consequence, many (but not all) questions about direct integrals can easily be answered without recourse to complicated measurability arguments. A sample question: Is the direct integral of quasidiagonal operators quasidiagonal? (A quasidiagonal operator [4] is one that is a limit of direct sums of finite matrices.) The affirmative answer to this question is a direct consequence of Theorem A (see Corollary B); a proof using more standard techniques (if one exists) would be a measure-theoretic nightmare.

Throughout, H denotes a separable Hilbert space and $(X, \mathscr{M}, \mu)$ denotes a σ-finite measure space such that $L^2(\mu)$ is a separable Hilbert space.

Let $L^2(\mu; H)$ denote (the equivalence classes of) all functions $f: X \to H$ such that f is Borel measurable and $\int_X \|f(x)\|^2 d\mu(x)$ is finite. This space is also denoted by $\int_X^\oplus H\, d\mu(x)$ and is called the *direct integral* of H over X. It is proved in [1] that $L^2(\mu; H)$ is a Hilbert space with the inner product defined by

$$(f, g) = \int_X (f(x), g(x))\, d\mu(x).$$

Let B(H) denote the set of *operators* (bounded linear transformations) on H. Let $L^\infty(\mu; B(H))$ denote the set of all functions $\tau: X \to B(H)$ such that τ is weakly

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Borel measurable (i.e., Borel measurable with respect to the weak operator topology) and essentially bounded (i.e., the mapping $x \mapsto \|\tau(x)\|$ is essentially bounded.) Each τ in $L^\infty(\mu; B(H))$ defines an operator T_τ on $L^2(\mu;H)$ by

$$(T_\tau f)(x) = \tau(x)f(x) \qquad \text{for every } x \in X.$$

The norm of T_τ is the essential supremum of the mapping $x \mapsto \|\tau(x)\|$. The operator T_τ is called the *direct integral* of $\tau(x)$ over X and is also denoted by $\int_X^\oplus \tau(x)\, d\mu(x)$. (There are more general direct integrals than the ones presented here, but the more general ones are unitarily equivalent to direct sums of the ones presented here [1].)

Two operators S,T are *approximately equivalent* if S is unitarily equivalent to arbitrarily small compact perturbations of T (i.e., there is a sequence $\{U_n\}$ of unitary operators such that $U_n^* SU_n - T$ is compact for $n = 1,2, \ldots$, and

$$\|U_n^* SU_n - T\| \to 0.$$

The proof of Theorem A is based mainly on a deep result of D. Voiculescu [5] on approximate equivalence (see also [3]).

THEOREM A. *For every* τ *in* $L^\infty(\mu;B(H))$ *there are distinct points* $x_1, x_2, \ldots$ *in* X *such that* $\int_X^\oplus \tau(x)\, d\mu(x)$ *is approximately equivalent to* $\tau(x_1) \oplus \tau(x_2) \oplus \cdots$.

COROLLARY B. *If a property of operators is preserved under direct sums and approximate equivalence, then it is preserved under direct integrals.*

COROLLARY C. *If a property of operators is preserved under approximate equivalence and under restrictions to reducing subspaces, then a direct integral* $\int_X^\oplus \tau(x)\, d\mu(x)$ *has the property only if* $\tau(x)$ *has the property almost everywhere.*

2. PRELIMINARIES

If $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H and $\{f_1, f_2, \ldots\}$ is an orthonormal basis for $L^2(\mu)$, then the set of functions ϕ_{ij} on X defined by $\phi_{ij}(x) = f_i(x)e_j$ is an orthonormal basis for $L^2(\mu;H)$. Thus $L^2(\mu;H)$ can be viewed as the tensor product of H and $L^2(\mu)$. This tensor product view makes the following lemma obvious.

LEMMA D. *If* $\tau \in L^\infty(\mu;B(H))$ *and* $S \in B(H)$ *and* $\tau(x) = S$ *for all* x *in* X, *then* T_τ *is unitarily equivalent to the direct sum of dim* $L^2(\mu)$ *copies of* S.

If $X_1, X_2, \ldots$ is a measurable partition of X, then $L^2(\mu;H)$ can naturally be identified with the direct sum $L^2(\mu_1; H) \oplus L^2(\mu_2; H) \oplus \ldots$, where μ_n is defined for each positive integer n by $\mu_n(E) = \mu(E \cap X_n)$. Similarly we can write

$$\int_X^\oplus \tau(x)\, d\mu(x) = \int_{X_1}^\oplus \tau(x)\, d\mu(x) \oplus \int_{X_2}^\oplus \tau(x)\, d\mu(x) \oplus \cdots$$

for each τ in $L^\infty(\mu; B(H))$. It follows that if $\tau \in L^\infty(\mu; B(H))$ and τ has countable range, then T_τ is unitarily equivalent to a direct sum of operators in the range of τ.

The next proposition is not difficult, but its statement and the example that follows help to put Theorem A into proper perspective. The simple proof is omitted. Note the identification of τ and T_τ.

PROPOSITION E. *Suppose* $\tau \in L^\infty(\mu; B(H))$. *Then*

(1) τ *is a (norm) limit of simple functions in* $L^\infty(\mu; B(H))$ *if and only if there is a subset* E *of* X *with* $\mu(E) = 0$ *such that* $\tau(X - E)$ *is a totally bounded subset of* B(H),

(2) τ *is a (norm) limit of functions in* $L^\infty(\mu; B(H))$ *having countable range if and only if there is a subset* E *of* X *with* $\mu(E) = 0$ *such that* $\tau(X - E)$ *is a (norm) separable subset of* B(H).

It follows that if H is finite-dimensional, then every τ in $L^\infty(\mu; B(H))$ is a limit of simple functions. If H is infinite-dimensional, let $\mathscr{K}(H)$ denote the set of compact operators on H. Since H is separable, then $\mathscr{K}(H)$ is separable. Hence a function τ in $L^\infty(\mu; B(H))$ is a limit of functions having countable range whenever $\tau(X) \subseteq \mathscr{K}(H)$.

The following example illustrates the difference between Theorem A and Proposition E. Let $X = [0,1]$, μ be Lebesgue measure, and let $H = L^2(\mu)$. Define $\tau: X \to B(H)$ by $\tau(x)f = \chi_{[0,x)} f$ for all f in $L^2(\mu)$ (where χ_E denotes the characteristic function of E). Since τ is weakly continuous, it follows that τ must be weakly Borel measurable. However, if $s \neq t$, then $\|\tau(s) - \tau(t)\| = 1$. Hence there is no set E of measure zero such that $\tau(X - E)$ is separable. Thus τ cannot be approximated by functions with countable range. On the other hand, T_τ is a projection with infinite rank and nullity, and T_τ is unitarily equivalent to every countable direct sum of the form $\tau(x_1) \oplus \tau(x_2) \oplus \cdots$, with $x_1, x_2, \ldots$ in (0,1).

3. PROOF OF THEOREM A

We first need the following result of Voiculescu [5]. Here $C^*(T)$ denotes the C^*-algebra generated by 1,T.

LEMMA F. *Suppose* $S,T \in B(H)$ *and* $C^*(S) \cap \mathscr{K}(H) = C^*(T) \cap \mathscr{K}(H) = 0$. *If there is a* $*$*-isomorphism between* $C^*(S)$ *and* $C^*(T)$ *that sends* S *onto* T, *then* S *and* T *are approximately equivalent.*

We also need a few facts concerning the $*$-strong operator topology on B(H). A net $\{T_i\}$ of operators converges $*$-strongly to an operator T if $T_i \to T$ and $T_i^* \to T^*$ strongly. For details concerning this topology the reader should consult [2]. All of the $*$-algebraic operations are sequentially $*$-strongly continuous. Also the set of weak Borel sets of operators coincides with the set of $*$-strong Borel sets. Furthermore, bounded subsets of B(H) are $*$-strongly separable and metrizable. Also the set of operators in B(H) that are images of an operator T under representations of $C^*(T)$ is $*$-strongly closed.

We now begin the proof. Let $T = \int_X^{\oplus} \tau(x)\, d\mu(x)$. If $E \subseteq X$, $\mu(E) > 0$, and $\tau|E$ is constant, then $\int_E^{\oplus} \tau(x)\, d\mu(x)$ is unitarily equivalent to a (finite or infinite) direct sum $\tau(x_1) \oplus \tau(x_2) \oplus \cdots$ for distinct points $x_1, x_2, \ldots$ in E. We can therefore assume that τ is never constant on sets with positive measure. In particular, this means that μ has no atoms. It follows that $L^\infty(\mu; B(H))$ contains no finite-rank projections. Since $C^*(T) \subseteq L^\infty(\mu; B(H))$, it follows that $C^*(T)$ contains no non-zero compact operators.

There is no harm in assuming that $\|\tau(x)\| \leqq 1$ for all x in X. The unit ball in B(H) is $*$-strongly separable and metrizable. Since τ is $*$-strongly Borel measurable and never constant on sets with positive measure, it follows that, after removing a set of measure zero if necessary, $\tau(X)$ has no $*$-strong isolated points. Since every separable metric space is Lindelöf, it follows that

$$\mu(\{x: \mu(\tau^{-1}(V)) = 0 \text{ for some } *\text{-strong neighborhood } V \text{ of } \tau(x)\}) = 0.$$

Hence we can assume that if $x \in X$ and V is a $*$-strong neighborhood of $\tau(x)$, then $\mu(\tau^{-1}(V)) > 0$.

Choose distinct points $x_1, x_2, \ldots$ in X so that $\{\tau(x_1), \tau(x_2), \ldots\}$ is $*$-strongly dense in $\tau(X)$, and let $S = \tau(x_1) \oplus \tau(x_2) \oplus \cdots$. Since $\tau(X)$ has no $*$-strong isolated points, it follows that $\{\tau(x_n), \tau(x_{n+1}), \ldots\}$ is $*$-strongly dense in $\tau(X)$ for $n = 1, 2, \ldots$. Thus the identity mapping of $C^*(S)$ can be factored (by restriction) through

$$C^*(\tau(x_n) \oplus \tau(x_{n+1}) \oplus \cdots) \qquad \text{for } n = 1, 2, \ldots.$$

Hence $C^*(S)$ contains no compact operators.

Suppose that p(x,y) is a non-commutative polynomial. It follows from the $*$-strong density of $\{\tau(x_1), \tau(x_2), \ldots\}$ in $\tau(X)$ and the $*$-strong semicontinuity of the operator norm that $\|p(S, S^*)\| = \sup \|p(\tau(x_n), \tau(x_n)^*)\| \geqq \|p(T, T^*)\|$.

On the other hand if n is a positive integer and $0 < r < \|p(\tau(x_n), (\tau(x_n)^*)\|$, then $V = \{W \in B(H): \|p(W, W^*)\| > r\}$ is a $*$-strong neighborhood of $\tau(x_n)$. Hence $\mu(\tau^{-1}(V)) > 0$. Therefore $\|p(T, T^*)\| > r$. It follows that

$$\|p(T, T^*)\| = \sup \|p(\tau(x_n), \tau(x_n)^*)\| = \|p(S, S^*)\|.$$

Since the non-commutative polynomials in an operator and its adjoint are dense in the C*-algebra it generates, it follows that there is a $*$-isomorphism from $C^*(S)$ to $C^*(T)$ that sends S onto T. Since $C^*(S)$ and $C^*(T)$ contain no non-zero compact operators, it follows from Lemma F that S and T are approximately equivalent.

Note that Corollary B follows immediately from Theorem A, but Corollary C follows from the fact that once the sets of positive measure on which τ is constant and an appropriate set of measure zero are removed, then $x_1, x_2, \ldots$ can be chosen to be an *arbitrary* sequence in X such that $\tau(x_1), \tau(x_2), \ldots$, is $*$-strongly dense in $\tau(X)$.

4. DIRECT INTEGRALS OF REPRESENTATIONS

Suppose $\mathfrak{A}$ is a separable, unital C*-algebra, and let $\mathrm{Rep}(\mathfrak{A}, B(H))$ denote the set of unital representations from $\mathfrak{A}$ into B(H). We topologize $\mathrm{Rep}(\mathfrak{A}, B(H))$ by saying that a net $\{\pi_n\}$ of representations of $\mathfrak{A}$ converges to a representation π if $\pi_n(a) \to \pi(a)$ $*$-strongly for each a in $\mathfrak{A}$. Choose a dense sequence $\{a_n\}$ in $\mathfrak{A}$ and a dense sequence $\{f_n\}$ in H. The topology on $\mathrm{Rep}(\mathfrak{A}, B(H))$ can be defined by the seminorms p_{ij}, $1 \leqq i,j < \infty$, where

$$p_{ij}(\pi) = \|\pi(a_i)f_j\| + \|\pi(a_i)^*f_j\| \qquad \text{for each } \pi \in \mathrm{Rep}(\mathfrak{A}, B(H)).$$

Therefore $\mathrm{Rep}(\mathfrak{A}, B(H))$ is a separable metric space.

If $(X,\mathscr{M},\mu)$ is a σ-finite measure space, $x \to \pi_x$ is a Borel measurable mapping from X into $\mathrm{Rep}(\mathfrak{A}, B(H))$. We define the direct integral $\int_X^{\oplus} \pi_x \, d\mu(x)$ as the representation on $\mathfrak{A}$ that sends each a in $\mathfrak{A}$ onto the operator $\int_X^{\oplus} \pi_x(a)\, d\mu(x)$ on $L^2(\mu;H)$.

Two representations π_1, π_2 are *approximately equivalent* if there is a sequence $\{U_n\}$ of unitary operators such that, for each a in $\mathfrak{A}$, we have $U_n^*\pi_1(a)U_n - \pi_2(a)$ is compact for each $n \geqq 1$ and $\|U_n^*\pi_1(a)U_n - \pi_2(a)\| \to 0$ as $n \to \infty$. Using Voiculescu's theorem (see Corollary 1.4 in [5]) it is an easy task to modify the proof of Theorem A to obtain a proof of the following theorem.

THEOREM G. *Every direct integral* $\int_X^{\oplus} \pi_x \, d\mu(x)$ *of representations in* $\mathrm{Rep}(\mathfrak{A}, B(H))$ *is approximately equivalent to a direct sum*

$$\pi_{x_1} \oplus \pi_{x_2} \oplus \cdots \qquad \text{for distinct points } x_1, x_2, \ldots \in X.$$

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