AN ARC IN A PL n-MANIFOLD WITH NO NEIGHBORHOOD THAT EMBEDS IN S^n, $n \geq 4$

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1. INTRODUCTION

To study the geometric embedding properties of a compactum X in a PL n-manifold, one sometimes finds it useful to change the scenery by reembedding a neighborhood of X in the n-sphere S^n. For example, if $n = 3$ and if X is a topological cell, this can always be done. (See [9] for a proof; the case of an arc was also done in [1].) The ability to make this transition in many instances greatly increases the usefulness of the "cellularity criterion" in 3-manifolds, for example. (See [9].)

It seems reasonable to expect that such familiar neighborhoods should be found whenever X is a "nice" compactum embedded in a PL n-manifold. Unfortunately, this is not so. It fails at the first opportunity: There is, for each $n \geq 4$, an arc A embedded in a certain PL n-manifold M^n such that no neighborhood of A in M^n embeds topologically in S^n. Our intuition is further violated by the fact that each proper subarc of A has a neighborhood that embeds in S^n.

Our basic four-dimensional construction is motivated by A. Casson's example of a cell-like continuum in a 4-manifold. Robert J. Daverman showed us this earlier construction, which seems by now to be widely known. It may be a counterexample to the four-dimensional cellularity criterion, at least in the smooth setting. No proof of this yet exists, however, and our methods definitely fail on it. We also wish to acknowledge that Robert D. Edwards suggested using decomposition space techniques (such as those of [5] and [12]) to shrink the Casson example down to an arc. Our overall approach owes much to these sources. We also thank Bob Sternfeld and Mike Starbird for many helpful discussions. In particular, Starbird showed us how to free our examples for $n > 4$ (Section 4) from relying on an as-yet-unpublished higher-dimensional version of the Edwards-Miller-Pixley-Eaton theorem.

Here is a brief summary of our notation. I denotes the unit interval [0,1]. B^n is $[-1,1]^n$; S^n is the n-sphere; and E^n is Euclidean n-space. A *loop in* X is a (continuous) mapping $S^1 \to X$. Integer coefficients are understood for (co) homology and $\tilde{H}$ denotes reduced (co) homology. Isomorphism of groups A, B is symbolized by $A \cong B$, and homeomorphism of spaces X, Y by $X \approx Y$. We work in the PL context throughout. A *cube-with-handles* is any PL homeomorph of the regular neighborhood in S^3 of a finite connected graph. Its *genus* is: one minus its Euler characteristic.

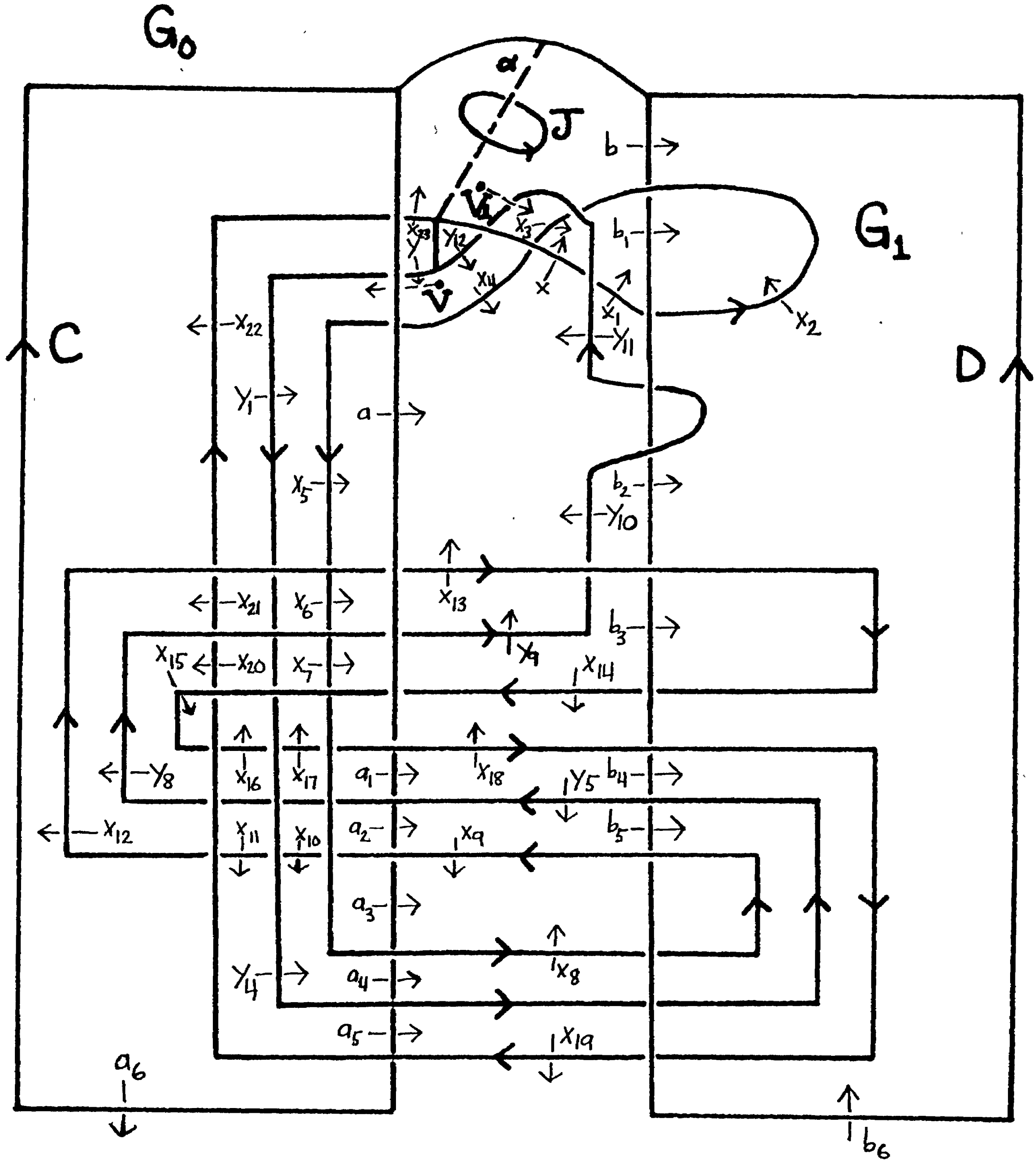
Received January 3, 1977.
Partially supported by a grant from the National Science Foundation.

Michigan Math. J. 25 (1978).

2. CONSTRUCTION OF A AND M^4

In this section we describe our example, fix some notation, and collect some elementary facts.

We first make a few remarks about our Figure, reference to which is made repeatedly throughout the paper. The Figure shows two disjoint, connected polyhedral graphs G_0, G_1 in S^3. Each of G_0, G_1 consists of two simple closed curves joined by an arc. The graphs G_0, G_1 are equivalently embedded as sets in S^3. There is also shown an arc α joining G_0 and G_1, and a simple closed curve J



"Generators for the fundamental group of S^3-G_0-G_1-α"

in $S^3 - G_0 - G_1 - \alpha$ that "locally links" α once. The Figure suggests in the usual way a presentation of $\pi_1(S^3 - G_0 - G_1 - \alpha)$, although for obvious reasons we have not written out the presentation in full. The basepoint is at the tip of the viewer's nose. The fifty-one generators shown are

$$\{a, a_1, \ldots, a_6, b, b_1, \ldots, b_6, x, x_1, \ldots, x_{23}, y, y_1, \ldots, y_{12}\},$$

and there is a relation for each singular point of the chosen projection of $G_0 \cup G_1 \cup \alpha$. We will take note of these relations among the generators only as we have need for them. Our groups are written multiplicatively. We use $\bar{a}$ for the inverse of a, and a^b for $\bar{b}ab$. The *commutator of* a *and* b, $[a,b]$, is $\bar{a}\bar{b}ab$.

The group $\pi_1(S^3 - G_0 - G_1)$ is obtained by "killing" J:

$$\pi_1(S^3 - G_0 - G_1) = \pi_1(S^3 - G_0 - G_1 - \alpha) / \langle \bar{x}y_{12}\bar{y}x_{23} \rangle ,$$

where $\langle R \rangle$ denotes the smallest normal subgroup of $\pi_1(S^3 - G_0 - G_1 - \alpha)$ containing R. The group $\pi_1(S^3 - G_1)$ is obtained by killing a and b:

$$\pi_1(S^3 - G_1) = \pi_1(S^3 - G_0 - G_1 - \alpha) / \langle a, b \rangle ,$$

and is, of course, free of rank two. To avoid a proliferation of notation, we will sometimes omit obvious inclusion homomorphisms, and will make slightly imprecise statements, such as: "In $\pi_1(S^3 - G_1)$, we have $y_{10} = y_{11}$, $x_1 = x_2$, $1 = a = a_1 = \ldots = a_6$, and $1 = b = b_1 = \ldots = b_6$."

Let H_1 be a thin regular neighborhood in S^3 of the graph G_1, and let H_0 be S^3 minus the interior of a thin regular neighborhood of G_0. Each H_i is a cube-with-handles of genus two, and our choices are such that $H_1 \subset \text{Int } H_0$. Each is standardly embedded in S^3. Choose an orientation-preserving homeomorphism $h: S^3 \to S^3$ that throws H_0 onto H_1. We have two *continua* (*i.e.*, compact, connected spaces) in S^3 of interest:

$$X_+ = \bigcap_{i=1}^{\infty} h^i(H_0), \quad \text{and}$$

$$X_- = \bigcap_{i=1}^{\infty} h^{-i}(S^3 - H_1),$$

where the exponents indicate iterated composition.

LEMMA 2.1. *For* $k \geq 1$ *and each* i, *the inclusion* $h^{i+k}(H_0) \to h^i(H_0)$ *induces the zero homomorphism on integral first homology.* (X_+ *is "strongly acyclic over the infinite cyclic group."*)

Proof. Consider first the inclusion $\alpha: H_1 \to H_0$. Let A,B be the natural free generators for $\pi_1(G_1)$ corresponding to its structure as an oriented graph, and let a,b be the free generators for $\pi_1(H_0)$ indicated in the Figure. These can be chosen so that $\alpha_*(A) = \bar{a}ba\bar{b} = [a,\bar{b}]$, and $\alpha_*(B) = b\bar{a}ba\bar{b}\bar{a}\bar{b}a = [\bar{b}, b\bar{a}\bar{b}a]$. Hence, $\alpha_*\pi_1(H_1)$ lies in the commutator subgroup of $\pi_1(H_0)$. Thus, α induces zero on integral first homology. For the general case, we have only to note that the inclusion $h^{i+k}(H_0) \to h^i(H_0)$ factors through the inclusion $h^{i+k}(H_0) \to h^{i+k-1}(H_0)$, which is topologically equivalent to α.

A continuum Y is *cell-like* if for some (and hence for every) embedding of Y in a compact ANR, we have: Each neighborhood U of Y contains a neighborhood V of Y such that V contracts to a point in U.

LEMMA 2.2. *For* $k \geq 0$ *and each* i, *the inclusion* $h^{i+k}(H_0) \to h^i(H_0)$ *induces a monomorphism on fundamental groups. Hence,* X_+ *is not cell-like.*

Proof. It suffices to consider $\alpha: H_1 \to H_0$. If α_* were not monic, then since $\pi_1(H_0)$ is free, $\alpha_*\pi_1(H_1)$ would be free of rank at most one. In particular, $\alpha_*\pi_1(H_1)$ would be abelian. However,

$$\alpha_*(A)\alpha_*(B) = \bar{a}b^2a\bar{b}\bar{a}\bar{b}a \neq b\bar{a}bab̄^2 = \alpha_*(B)\alpha_*(A)$$

in $\pi_1(H_0)$.

LEMMA 2.3. *For* $k \geq 1$ *and each* i, *the inclusion*

$$h^i(S^3 - H_1) \to h^{i+k}(S^3 - H_1)$$

induces the zero homomorphism on fundamental groups. Hence, $h^i(S^3 - H_1)$ *contracts to a point in* $h^{i+k}(S^3 - H_1)$, *and* X_- *is cell-like.*

Proof. We consider first the inclusion $h^{-1}(S^3 - H_1) \to S^3 - H_1$. (Or, equivalently for our purposes, we consider the inclusion $G_0 \to S^3 - G_1$.) Let C,D be the free generators for $\pi_1(G_0)$ shown in the Figure. When C and D are homotoped slightly off G_0 and considered as elements in $\pi_1(S^3 - G_0 - G_1 - \alpha)$, we obtain

$$\{C\} = \bar{x}_{18}y_5x_9\bar{x}_8\bar{y}_4x_{19}, \quad \text{and} \quad \{D\} = x_2\bar{y}_{10}\bar{x}_{13}\bar{x}_{18}y_5x_9 .$$

Adding the relations "$a = 1 = b$" to $\pi_1(S^3 - G_0 - G_1 - \alpha)$ yields $\pi_1(S^3 - G_1)$. In $\pi_1(S^3 - G_1)$, we find (with a slight abuse of notation) that $x_9 = x_8$, $y_5 = y_4$, and $x_{18} = x_{19}$, so that $\{C\} = 1$. To show that $\{D\} = 1$ when $\{D\}$ is considered in $\pi_1(S^3 - G_1)$, we read the following facts from the Figure (assuming $a = 1 = b$):

$$y_{10} = y_{11},\ x_1 = x_2,\ x_{12} = x_{13} = x_{14} = x_{15},$$

$$y_8 = y_9 = x_{13}y_{10}\bar{x}_{13},$$

$$x_2 = y_{10}x\bar{y}_{10},$$

$$M_1 = \bar{x}_{12}\bar{y}_8x_{15} = \bar{x}_{13}(x_{13}\bar{y}_{10}\bar{x}_{13})x_{15} = \bar{y}_{10},$$

$$M_2 = \bar{x}_8\bar{y}_4x_{19} = M_2M_1\bar{M}_2 \Rightarrow M_1 = M_2 = \bar{y}_{10},$$

$$x_{13} = \bar{M}_2\bar{M}_1 x M_1M_2 = y_{10}{}^2x\bar{y}_{10}{}^2,$$

$$\bar{x}_{18}y_5x_9 = M_2(\bar{x}_{14}y_9x_{13})\bar{M}_2 = \bar{y}_{10}(\bar{x}_{13}y_9x_{13})y_{10} = \bar{y}_{10}y_{10}y_{10} = y_{10}.$$

(M_1 and M_2 are *defined* by the expressions immediately to their right.) Hence,

$$(x_2)(\bar{y}_{10})(\bar{x}_{13})(\bar{x}_{18}y_5x_9) = (y_{10}x\bar{y}_{10})(\bar{y}_{10})(y_{10}{}^2\bar{x}\bar{y}_{10}{}^2)(y_{10}) = 1,$$

as desired. The general result claimed now follows from an argument like that completing Lemma 2.1.

LEMMA 2.4. $S^3 - X_+$ *is contractible. In fact, the suspension* $\sum X_+$ *is* PL *cellular in* $\sum S^3 = S^4$.

Proof. Each compact set in $S^3 - X_+$ lies in $h^i(S^3 - H_1)$ for some i, and hence contracts to a point in $h^{i+1}(S^3 - H_1)$. Hence, $S^3 - X_+$ is contractible. To show that $\sum X_+$ is PL cellular in S^4, it suffices to show that $S^4 - \sum X_+$ is PL homeomorphic to E^4. But $S^4 - \sum X_+$ is PL homeomorphic to $(S^3 - X_+) \times E^1$, so the result follows from Theorem 2 of [8].

Recall that X_+, X_- are in $S^3 = \partial B^4$, and that $B^4 = [-1, 1]^4$. Identify B^4 with $B^4 \times \{0\}$. Let M^4 be the PL 4-manifold obtained from the three disjoint 4-cells $B^4 \times \{-1, 0, 1\}$ by the identifications

$$(x, 1) \equiv (x, 0) \qquad \text{for every } x \in H_1,$$
$$\text{and } (x, -1) \equiv (x, 0) \qquad \text{for every } x \in S^3 - \text{Int } H_0.$$

If $S \subset S^3$ and $i \in \{-1,0,1\}$, let $C_i(S)$ be obtained by coning over $S \times \{i\}$ from the origin in $B^4 \times \{i\}$. We consider the suspension $\sum X_+ \subset M^4$ as

$$\sum X_+ = C_0(X_+) \cup C_1(X_+),$$

and $\sum X_- \subset M^4$ as $\sum X_- = C_0(X_-) \cup C_{-1}(X_-)$. Then we define X as the one-point union, or "wedge": $X = \sum X_+ \cup \sum X_- \subset M^4$.

LEMMA 2.5. X *is a cell-like continuum in the simply connected* PL *4-manifold* M^4. *Further,* $\sum X_+$ *is* PL *cellular in* M^4, *and* $\sum X_-$ *also has a neighborhood in* M^4 *that* PL *embeds in* S^4.

Proof. Each of $\sum X_+$ and $\sum X_-$ is a cell-like continuum by Theorem 3 of [10]. The wedge X of two cell-like continua is easily shown to be a cell-like continuum. Simple connectivity of M^4 follows from van Kampen's Theorem. It has previously been remarked that $\sum X_+$ is PL cellular in $S^4 = \sum S^3$. Since there is a natural mapping of $B^4 \times \{0, 1\} \subset M^4$ into $\sum S^3$ that completes the identification of $S^3 \times \{0\}$ with $S^3 \times \{1\}$ and is the identity on a neighborhood in M^4 of $\sum X_+$, it follows that $\sum X_+$ is also PL cellular in M^4. Similarly, $\sum X_-$ has a neighborhood in M^4 that PL embeds in S^4.

Notation. There is a natural sequence of "describing" neighborhoods for X in M^4. For each $i \geq 1$, let $N_i \subset M^4$ consist of:

$$C_1(h^i(H_0)) \cup C_{-1}(h^{-i}(S^3 - H_1)) \cup C_0(h^i(H_0) \cup h^{-i}(S^3 - H_1)) \cup B^4(i) \times \{-1, 0, 1\},$$

where $B^4(i)$ is the 4-cell $[-1/i, 1/i]^4$. We have:

LEMMA 2.6. *Each* N_i *is a compact, simply connected,* PL *4-manifold which (to within* PL *homeomorphism) can be obtained from* $B^4 \times \{0\}$ *by pasting on 4-cells* $B^4 \times \{1\}$ *and* $B^4 \times \{-1\}$ *along* $h^i(H_0)$ *and* $h^{-i}(S^3 - H_1)$, *respectively, via the "identity" mapping. The* N_i*'s form a neighborhood basis for* X *in* M^4.

Since $\sum X_+$ is cellular in M^4, there is by [2] a mapping $f: M^4 \to M^4$ whose restriction to ∂M^4 is the identity, and whose only nondegenerate point-inverse is $\sum X_+$. Note that f(X) is topologically $\sum X_-$. Consider the upper semicontinuous decomposition G_1 of M^4 whose only nondegenerate elements are the f-images of the disjoint copies of X_- at the different levels of $\sum X_-$. By [5], [12], or [4], some mapping $g: M^4 \to M^4$ restricts to the identity on ∂M^4 and has exactly the elements of G_1 for its point-inverses. The arc mentioned in the Introduction is $g(f(X)) = A$.

LEMMA 2.7. *For each neighborhood* U *of* X *in* M^4, g(f(U)) *is homeomorphic to* U.

Proof. By [2], some mapping $\bar{f}$ of U onto U restricts to the identity off a compact neighborhood of $\sum X_+$ in the interior of U, and has $\sum X_+$ as its only nondegenerate point-inverse. By [5], [12], or [4], some mapping $\bar{g}$ of U onto U has exactly the $\bar{f}$-images of the disjoint copies of X_- at the different levels of $\sum X_-$ for its nondegenerate point-inverses. Then $\bar{g}\bar{f}f^{-1}g^{-1}$ is a homeomorphism of g(f(U)) onto U.

3. NEIGHBORHOODS OF A IN M^4

Throughout this section, let A, M^4, N_i, etc. be as constructed in Section 2.

Following [13], we say that compact sets X,Y in S^n are I-*equivalent* if for some $Z \subset S^n \times [0, 1]$, $Z \cap S^n \times \{0\} = X \times \{0\}$ and $Z \cap S^n \times \{1\} = Y \times \{1\}$, and some homeomorphism of Z with $X \times [0,1]$ carries $X \times \{0\}$ to $X \times \{0\}$ and $Y \times \{1\}$ to $X \times \{1\}$.

Recall that if G is a group and S_1, S_2 are nonempty subsets of G, then $[S_1,S_2]$ denotes the subgroup of G generated by all *commutators* $[x,y] = \bar{x}\bar{y}xy$, where $x \in S_1$ and $y \in S_2$. The *lower central series* of G is defined thus:

$$G_1 = G, \quad G_{n+1} = [G_n, G] \qquad \text{for } n \geq 1,$$

and $G_\omega = \bigcap \{G_n: n = 1, 2, \ldots\}$. An element of G_ω is called an *omegatator.* We say that $x \equiv y \pmod{G_n}$ if $x\bar{y} \in G_n$.

In Theorem 5.2 of [13], we find a necessary condition for I-equivalence:

PROPOSITION (Stallings [13]). *Let* X *and* Y *be* I-*equivalent compact sets in* S^n. *Let* $A = \pi_1(S^n - X)$ *and* $B = \pi_1(S^n - Y)$. *Then for each finite* k *the lower central quotient groups* A/A_k *and* B/B_k *are isomorphic.*

The object of the next two lemmas is to develop a test for I-equivalence of X and Y, when X is a finite graph of two components.

LEMMA 3.1. *Let* $M^3 \subset S^3$ *be a compact* 3-*manifold with nonempty, connected boundary of genus* n. *Let* $\Gamma \subset M^3$ *be a connected graph of Euler characteristic* $1-n$ *whose inclusion into* M^3 *induces a surjection on integral first homology. Let* $F = \pi_1(\Gamma)$, *a free group of rank* n, *and let* $G = \pi_1(M^3)$. *Then, for each finite* k, *the inclusion* $\Gamma \subset M^3$ *induces an isomorphism* $F/F_k \cong G/G_k$.

Proof. By Alexander Duality and the Mayer-Vietoris Theorem, $H_1(M^3)$ ($\cong G/G_2$) is free abelian of rank n. Hence, the inclusion $\Gamma \subset M^3$ induces an isomorphism on integral first homology. Since $H_2(M^3) = 0$, the result follows from Theorem 5.1 of [13].

LEMMA 3.2. *Let* $M^3 \subset S^3$ *be a compact* 3-*manifold whose boundary has exactly two components. Suppose that the sum of the genera of these two components is* n, *and let* α *be an arc in* M^3 *joining these components, with* $\alpha \cap \partial M^3 = \partial\alpha$. *Suppose that* J *is a simple closed curve bounding a* 2-*cell in* Int M^3 *that intersects* α *transversely at a single point. Let* G *denote* $\pi_1(M^3)$, *and* F *denote a free group of rank* n. *Then,* $F/F_k \cong G/G_k$ *for each finite* k, *if and only if* J *represents an element of the* ω^{th} *term of the lower central series of* $\pi_1(M^3 - \alpha)$.

Proof. Consider the inclusion-induced surjection

$$K = \pi_1(M^3 - \alpha) \to \pi_1(M^3) = G,$$

whose kernel is precisely N, the normal closure of {J} in K. By Lemma 3.1, applied to M^3 minus a regular neighborhood of α, $F/F_k \cong K/K_k$ for each finite k.

Suppose first that $F/F_k \cong G/G_k$ for each k. Then for each k we have a surjection $K/K_k \to G/G_k$ of groups each isomorphic to F/F_k. But F/F_k is *Hopfian,* by Theorem 5.5 of [7]. This implies that the surjection $K/K_k \to G/G_k$ is an isomorphism. Since J contracts to a point in M^3, J represents an element of K_k for each k, as claimed.

On the other hand, if J represents an element of K_ω, then the inclusion-induced homomorphism $K/K_k \to G/G_k$ is an isomorphism for each k. Hence, $G/G_k \cong F/F_k$ for each k. The proof is complete.

If G is a group and $z \in G$, then the *weight of* z *in* G, $\omega(z,G)$, is defined to be the largest integer $n \geq 1$ for which $z \in G_n$. If no largest such integer exists, we put $\omega(z,G) = \infty$. Clearly, $\omega(z,G) \geq n$ if and only if $z \in G_n$. We will shorten "$\omega(z,G)$" to "$\omega(z)$" when there is no confusion as to which group G is being considered.

The next lemma follows from Corollary 5.12 (iii), page 342 of [7].

LEMMA 3.3 *Let* F *be a free group of finite rank, and let* U,V *be elements of finite weight in* F. *Suppose that either:* $\omega(U) \neq \omega(V)$; *or* $\omega(U) = m = \omega(V)$ *and that* U,V *fail to determine powers of the same element in* F_m/F_{m+1}. *Then,*

$$\omega([U, V]) = \omega(U) + \omega(V).$$

The next lemma is not absolutely necessary for our proof, but we quote it at one point to explain our strategy.

LEMMA 3.4. *Let* G *be a group, and let* U,V *be elements of different finite weights in* G. *Then,* $\omega(UV) = \min\{\omega(U), \omega(V)\}$.

Proof. Let $u = \omega(U)$ and $v = \omega(V)$, where $u < v$. Clearly, $UV \in G_u$. On the other hand, since $U = (UV)\bar{V}$ and since G_{u+1} is a subgroup containing $\bar{V}$ but not U, UV cannot belong to G_{u+1}. Hence, $\omega(UV) = u$.

We are now ready for our main result:

THEOREM 1. *No neighborhood of* A *in* M^4 *embeds topologically in* S^4.

Proof. By Lemma 2.7, it suffices to show that none of the neighborhoods N_i of X in M^4 embeds in S^4. Fix the integer $i \geq 1$. Using the notation of Section 2, there is a PL 4-cell $D^4 \subset \text{Int } N_i$ obtained by piping $B^4(i+1) \times \{-1\}$ to

$$B^4(i+1) \times \{1\}$$

via a tube ($\equiv$ PL 4-cell) that lies close to, but misses,

$$N_{i+1} - (B^4(i+1) \times \{-1, 1\}).$$

The result is that $N_{i+1} \cap D^4 = B^4(i+1) \times \{-1, 1\}$, and that the pair $(\partial D^4, N_{i+1} \cap \partial D^4)$ is homeomorphic to the pair whose first entry is S^3 and whose second entry consists of two disjoint, standardly embedded cubes-with-handles K_1, K_2 of genus two that can be separated by a 2-sphere in S^3.

Now suppose that $h:N_i \to S^4$ is an embedding. Then, using [2] and the fact that each of D^4 and $B^4(i+1) \times \{0\}$ is PL relative to the structure on N_i,

$$S^4 - h(\text{Int } D^4) - h(\text{Int } B^4(i+1) \times \{0\})$$

is topologically $S^3 \times [0,1]$. Hence, the set

$$h^{i+1}(H_0) \cup h^{-i-1}(S^3 - H_1) \subset S^3$$

is I-equivalent to the set $K_1 \cup K_2$. Thus, if F is free of rank four, and

$$G = \pi_1(S^3 - h^{i+1}(H_0) - h^{-i-1}(S^3 - H_1)),$$

then $G/G_k \cong F/F_k$ for each k, by Stallings' Proposition. We will show this last conclusion to be false.

The 3-manifold $S^3 - h^{i+1}(\mathrm{Int}\, H_0) - h^{-i-1}(S^3 - \mathrm{Int}\, H_1)$ is homeomorphic to

$$M^3 = S^3 - \mathrm{Int}\, H_1 - h^{-2i-1}(S^3 - \mathrm{Int}\, H_1) = h^{-2i}(H_0) - \mathrm{Int}\, H_1,$$

via h^{-i}. Let α be an arc in M^3 joining the two components of ∂M^3, such that $\alpha \cap \partial M^3 = \partial\alpha$, and $\alpha \cap h^{-k}(\partial H_1)$ is a single point for $k = 0, 1, \ldots, 2i + 1$. Put

$$H_0 - \mathrm{Int}\, H_1 = M_0{}^3 \subset M^3.$$

Let $J \subset M_0{}^3 - \alpha$ bound a 2-cell in Int $M_0{}^3$ that intersects α transversely at a single point. By Lemma 3.2, it suffices to show that J does not represent an element of K_ω, where $K = \pi_1(M^3 - \alpha)$.

Refer to the Figure, which shows α,J and the usual Wirtinger-presentation generators for $\pi_1(S^3 - G_0 - G_1 - \alpha) \cong \pi_1(M_0{}^3 - \alpha) = K_0$. Two important loops V,$V_1$ in ∂H_1 are partially shown in the Figure. Loop V starts at the basepoint, goes straight to the dot over the letter V in the upper middle part of the Figure, and then follows closely the indicated simple closed curve in G_1, staying always on the same "side" of this curve with the exception of two twists near the end of its journey corresponding to the generators y_{10} and y_{11}. The complete (fourteen-letter) V-word is

$$\{V\} = \bar{a}\bar{x}_{12}\bar{y}_8x_{15}b_6\bar{x}_8\bar{y}_4x_{19}ax_{13}y_{10}\bar{b}_1y_{11}\bar{x}\,.$$

Similar remarks apply to the loop V_1, which follows closely the other simple closed curve of G_1. Its only unexpected twist occurs at the beginning of its journey and corresponds to the generator x. The complete (twenty-four-letter) V_1 word is

$$\{V_1\} = x\bar{y}_{11}b_1y_{11}\bar{x}\bar{a}\bar{x}_{12}\bar{y}_8x_{15}b_6\bar{x}_8\bar{y}_4x_{19}a\bar{b}_3\bar{a}\,\bar{x}_{19}y_4x_8\bar{b}_6\bar{x}_{15}y_8x_{12}a.$$

Some straightforward calculation yields the statement

$$(*) \qquad \{V_1\} = [\bar{b}_2{}^{y_{10}b_1y_{11}\bar{x}}, \{\bar{V}\}]$$

in K_0. (Write out the right hand side of (*), using the earlier expression for {V}, and making the substitutions $b_2 = y_{10}b_1\bar{y}_{10}$ and $\bar{b}_2 = \bar{x}_{13}\bar{b}_3x_{13}$. The result is the earlier expression for $\{V_1\}$.)

Since J bounds an orientable, punctured surface of genus two in $M_0{}^3 - \alpha$, {J} represents (up to conjugacy) $[V_1, \bar{x}]\,[V, \bar{y}] = L$ in K_0. (One could also verify this by showing that $L = x_{23}\bar{x}y_{12}\bar{y}$, and noting that this last expression represents the conjugacy class of {J}.) The proof will be completed as follows. It will be shown first that $\{V_1\}$ and {V}, when considered in K, have different finite weights. Then, by Lemmas 3.1 and 3.3, it will follow that $[V_1, \bar{x}]$ and $[V, \bar{y}]$ have different finite weights in K. Finally, Lemma 3.4 will show that L has finite weight in K, as desired.

To accomplish this program, we first note that by Lemma 2.2, the inclusion $H_1 \to h^{-2i-1}(H_1)$ induces a monomorphism on fundamental groups. Hence, since neither of V, V_1 contracts in H_1, neither contracts in $h^{-2i-1}(H_1)$. Thus, V, V_1 represent elements of finite weight in the free group $\pi_1(h^{-2i-1}(H_1))$. Thus, since

$$M^3 - \alpha \subset h^{-2i-1}(H_1),$$

each of $\{V\}$, $\{V_1\}$ has finite weight when considered in K. The displayed equation (*) is valid when its members are considered in K. Hence,

$$1 < \omega(\{V\}, K) < \omega(\{V_1\}, K) = w < \infty.$$

Let Γ be a finite, connected graph in $M^3 - \alpha$ with Euler characteristic -3, with some based loop c in Γ homotopic to x, some based loop d in Γ homotopic to y, and such that the inclusion $\Gamma \to M^3 - \alpha$ induces a surjection on first homology. Then, by Lemma 3.1 (with $n = 4$ and applied to the 3-manifold: M^3 minus a regular neighborhood of α), we have that $\pi_1(\Gamma)/\pi_1(\Gamma)_k \to K/K_k$ is an isomorphism for each finite k. In other words, the inclusion-induced homomorphism ϕ: $\pi_1(\Gamma) \to K$ is *weight-preserving* in the sense that $\omega(z, \pi_1(\Gamma)) = \omega(\phi(z), K)$ for each $z \in \pi_1(\Gamma)$.

For some elements W, W_1 in $\pi_1(\Gamma)$,

$$\phi(W_1) \equiv \{V_1\} \pmod{K_{w+1}}, \qquad \text{and} \qquad \phi(W) \equiv \{V\} \pmod{K_{w+1}}.$$

Since also $1 < \omega(\{V\}, K) < w$, we have $1 < \omega(W, \pi_1(\Gamma)) < w$. But $\omega(\bar{d}, \pi_1(\Gamma)) = 1$, so by Lemma 3.3 $\omega([W, \bar{d}]) \leq w$, and hence $\omega([\{V\}, \bar{y}]) \leq w$. Thus mod K_{w+1}

$$\{J\} \equiv [\{V_1\}, \bar{x}]\,[\{V\}, \bar{y}] \equiv [\{V\}, \bar{y}] \not\equiv 0.$$

This shows that $\omega(\{J\}, K) \leq w$, and hence $\{J\} \notin K_\omega$. The proof is complete.

We conclude with a general result concerning the situation considered in the proof of our example. It requires less information in its hypothesis than we had in our example, and also yields a weaker conclusion. That is, there are compact 3-manifold groups G with G/G_ω not free, and yet $G/G_k \cong F/F_k$ for the appropriate free group F and each finite k.

Notation. If S is a nonempty set of loops in an arcwise-connected space X, we let N(S;X) denote the smallest normal subgroup of $\pi_1(X)$ containing the conjugate class in $\pi_1(X)$ determined by each of the loops in S.

THEOREM 2. *Let* H_0 *and* H_1 *be* PL 3-*manifolds, where* $\pi_1(H_0)_\omega = \{1\}$, H_1 *is a cube-with-handles, and the inclusion* $H_1 \subset \operatorname{Int} H_0$ *induces a monomorphism on fundamental groups. Choose a complete set of meridional disks* $\{D_1, \ldots, D_n\}$ *for* H_1 $(n \geq 1)$ *and a basepoint for fundamental groups in* ∂H_1. *If* i *denotes the inclusion* $\partial H_1 \to H_0 - \operatorname{Int} H_1 = M^3$, *and* G *denotes* $\pi_1(M^3)$, *then we have*

$$i_*^{-1}(G_\omega) \subset i_*^{-1}\left(N\left(\bigcup \partial D_m; M^3\right)\right) = N\left(\bigcup \partial D_m; \partial H_1\right).$$

Further, if each closed surface separates H_0, *then* $\pi_1(\partial H_1)/i_*^{-1}(G_\omega)$, *and hence also* G/G_ω, *fails to be a free group.*

Proof. Let $D^* = \bigcup D_m$ and $\partial D^* = \bigcup \partial D_m$. By attaching 2-handles with the D_m's as cores to ∂M^3, we see that $G/N(\partial D^*; M^3) \cong \pi_1(H_0)$, an omegatatorless group. Thus, $G_\omega \subset N(\partial D^*; M^3)$, and so $i_*^{-1}(G_\omega) \subset i_*^{-1}N(\partial D^*; M^3)$.

To see the inclusion $i_*^{-1}(N(\partial D^*; M^3)) \subset N(\partial D^*; \partial H_1)$, let $W \subset \partial H_1$ be a wedge at the basepoint of n simple closed curves naturally induced by the D_m's. That is, each $W \cap \partial D_m$ is a point of transverse intersection in ∂H_1, the correspondence $m \to W \cap \partial D_m$ is one-to-one, and $W - D^*$ is connected. In particular, H_1 is a regular neighborhood of $W \cup D^*$. Let $R: \partial H_1 \to W$ be the natural retraction, with $\operatorname{Ker} R_* = N(\partial D^*; \partial H_1)$. Let j be the inclusion $W \to H_0$ and k the inclusion $M^3 \to H_0$. Suppose $x \in \pi_1(\partial H_1)$ and $i_*(x) \in N(\partial D^*; M^3)$. Then $k_* i_*(x) = j_* R_*(x) = \{1\}$ in $\pi_1(H_0)$, so that since j_* is a monomorphism by hypothesis we have $R_*(x) = 1$ in $\pi_1(W)$, as desired to prove our first inclusion. The reverse inclusion is obvious, and yields equality of the two subgroups.

If, contrary to expectation, $\pi_1(\partial H_1)/i_*^{-1}(G_\omega) = Q$ is a free group, then by Corollary 3.3 of [6], its rank is $q, q \leq n$. But there is a surjection $Q \to \pi_1(\partial H_1)/N(\partial D^*; \partial H_1)$, and the quotient group on the right is free of rank n. Thus, $q = n$. It follows (Theorem 5.5 of [7]) that the above surjection is an isomorphism. Hence, we have $i_*^{-1}(G_\omega) = N(\partial D^*; \partial H_1)$. In particular, ∂D_1 represents an element of $G_\omega \subset G_2$. Hence, there is a compact, orientable 2-manifold $M^2 \subset M^3$ with $M^2 \cap \partial M^3 = \partial M^2 = \partial D_1$. But then $D_1 \cup M^2$ is a surface that fails to separate H_0, a contradiction.

Hence, Q fails to be a free group. Further, Q embeds in G/G_ω which therefore also fails to be free. The proof is complete.

4. EXAMPLES IN DIMENSIONS FIVE AND HIGHER

Let $E_+^n = \{(x_1, \ldots, x_n) \in E^n: x_n \geq 0\}$. The phrase "*spin* E_+^3 *about* E^2 *by* S^{n-4}", where $n \geq 5$, means: Take the quotient space of $E_+^3 \times S^{n-4}$ obtained by identifying each of the (n−4)-spheres $\{\{p\} \times S^{n-4}: p \in E^2\}$ to a single point. This quotient space is, of course, topologically E^{n-1}. If $X \subset E_+^3$, then by the *spin of* X, Sp (X), we understand the subset of this quotient space corresponding to $X \times S^{n-4}$. We will keep the dimension of the sphere by which we are spinning fixed during a given discussion.

For reference, we state a form of the main result of [5], [12], and [4]. Our version follows from the proofs to be found in these references. For our purposes, we say that an upper semicontinuous decomposition G of a space Y is *shrinkable* if for each open $U \subset Y$ containing the nondegenerate elements of G, there is a proper surjective mapping $f: Y \to Y$ such that $f | Y - U$ = identity and such that the point-inverses of f are precisely the elements of G.

THEOREM (Edwards-Miller [5], Pixley-Eaton [12]). *Let* $X \subset E_+^3 - E^2$ *be a cell-like continuum. Let* E^{n-1} *be obtained by spinning* E_+^3 *about* E^2 *by* S^{n-4}, *where* $n \geq 5$. *Let G be the upper semicontinuous decomposition of* E^{n-1} *whose only* *nondegenerate elements are the disjoint positions occupied by* X *during the spin:* $\{X \times \{t\} : t \in S^{n-4}\}$. *Then* G *is shrinkable. In particular,* $E^{n-1}/G \approx E^{n-1}$.

John L. Bryant showed in [3] that E^n modulo an (n−1)-cell is a factor of E^{n+1}. Hence:

COROLLARY. *Assuming the hypotheses and notation of the previous theorem, let* A *be an arc in* E_+^3 *joining* X *to* E^2 *and such that* $(E^2 \cup X) \cap \text{Int } A = \emptyset$. *Let* $f: E^{n-1} \to E^{n-1}$ *demonstrate that* $E^{n-1}/G \approx E^{n-1}$. *Let* H *be the upper semicontinuous decomposition of* $E^n = E^{n-1} \times E^1$ *whose only nondegenerate elements are the* (n−3)-*cells* $\{f(\text{Sp}(X \cup A)) \times \{s\} : s \in E^1\}$. *Then* H *is shrinkable. In particular,*

$$E^{n-1}/\text{Sp}(X \cup A) \times E^1 \approx E^n.$$

A continuum Y is *strongly acyclic* (over the infinite cyclic group $\mathbb{Z}$) if for some (and hence for every) embedding of Y in a compact ANR, we have: Each neighborhood U of Y contains a neighborhood V of Y such that the inclusion $V \to U$ induces zero on integral homology in each positive dimension. If Y is compact metric and finite-dimensional, then each of the statements "$\check{H}^*(Y;\mathbb{Z}) = 0$" and "the suspension of Y is cell-like" is equivalent to the strong acyclicity of Y. (See Theorem 3 of [10].) From the first statement and duality, we have:

LEMMA 4.1. *Let* Y *be a compact connected set in* E^n $(n \geq 3)$, *with* $E^n - Y$ *connected. Then,* Y *is strongly acyclic over* $\mathbb{Z}$ *if and only if*

$$H_i(E^n - Y) = 0 \qquad \textit{for } i = 1, 2, \ldots, n-2.$$

As an application, we have:

LEMMA 4.2. *Let* E^{n+2} *be obtained by spinning* E_+^3 *about* E^2 *by* S^{n-1} $(n \geq 2)$. *Suppose that some continuum* $X \subset E_+^3$ *is strongly acyclic, and that* $X \cap E^2$ *is a single point* p. *Then* $Y = \text{Sp}(X)$ *is strongly acyclic. Further, if* X *is cell-like, so also is* Y.

Proof. It is clear that each of Y and $E^{n+2} - Y \approx \text{Sp}(E_+^3 - X)$ is connected. Hence, by Lemma 4.1, it suffices to show that $H_i(E^{n+2} - Y) = 0$ for $i = 1, 2, \ldots, n$. Note that, since X is strongly acyclic and $X \cap E^2 = \{p\}$,

$$H_i(E_+^3 - X) = 0 \qquad \text{for } i > 0.$$

Thus, $H_*((E_+^3 - X) \times S^{n-1}) = H_*(S^{n-1})$.

Let N be a collar neighborhood of $E^2 - \{p\}$ in $E_+^3 - X$. (We assume that the thickness of N tends to zero near p.) Then, putting

$$\sum_1 = \text{Sp}(N) \quad \text{and} \quad \sum_2 = \text{Sp}(E_+^3 - E^2 - X),$$

we have $E^{n+2} - Y = \sum_1 \cup \sum_2$. Note that

$$H_*\left(\sum_1\right) = H_*(S^1),\quad H_*\left(\sum_2\right) = H_*(S^{n-1}),\quad H_*\left(\sum_1 \cap \sum_2\right) = H_*(S^1 \times S^{n-1}),$$

and that the inclusions $\sum_i \to E^{n+2} - Y$ (i=1,2) induce zero on homology.

We show by induction that for each i, $1 \le i \le n$, $H_i(E^{n+2} - Y) = 0$. Assuming that $\tilde{H}_{i-1}(E^{n+2} - Y) = 0$ for some i, $1 \le i \le n$, consider part of the natural Mayer-Vietoris sequence for $E^{n+2} - Y$:

$$\ldots \to H_i\left(\sum_1\right) \oplus H_i\left(\sum_2\right) \xrightarrow{\alpha} H_i(E^{n+2} - Y) \xrightarrow{\beta} H_{i-1}\left(\sum_1 \cap \sum_2\right) \xrightarrow{\gamma}$$
$$H_{i-1}\left(\sum_1\right) \oplus H_{i-1}\left(\sum_2\right) \to 0.$$

By earlier remarks, $\alpha = 0$, so that β is monic. Since $i \le n$,

$$H_{i-1}\left(\sum_1 \cap \sum_2\right) \cong H_{i-1}\left(\sum_1\right) \oplus H_{i-1}\left(\sum_2\right).$$

Since γ is a surjection between isomorphic, finitely-generated abelian groups, γ is monic. Hence $H_i(E^{n+2} - Y) = 0$, as desired.

Now suppose that X is cell-like, and let the neighborhood U of Y in E^{n+2} be given. Choose a neighborhood U_1 of X in E_+^3 such that $Sp(U_1) \subset U$. Since X is cell-like, there are compact neighborhoods V_0, V_1 of X in E_+^3 such that $V_1 \cap E^2$ is a 2-cell and each inclusion $V_1 \to V_0$, $V_0 \to U_1$ is homotopic to a constant. The Homotopy Extension Theorem yields a homotopy $h_t: V_1 \to U_1$ such that: h_0 = inclusion; h_1 retracts V_1 onto $V_1 \cap E^2$; and $h_t(x) = x$ for each $t \in I$ and $x \in V_1 \cap E^2$. We claim that $V = Sp(V_1)$ contracts to a point in $Sp(U_1) \subset U$. To verify this, note that $h_t \times$ (identity) is a deformation retraction of $V_1 \times S^{n-1}$ onto $(V_1 \cap E^2) \times S^{n-1}$ in $U_1 \times S^{n-1}$ that keeps $(V_1 \cap E^2) \times S^{n-1}$ pointwise fixed. Passing to the quotient space yields a deformation retraction of V onto

$$Sp(V_1 \cap E^2) \approx V_1 \cap E^2$$

in $Sp(U_1)$ that keeps $V_1 \cap E^2$ pointwise fixed. This completes the proof.

LEMMA 4.3. *Let* H_0, H_1 *be the polyhedral, equivalently embedded cubes-with-handles constructed earlier, with* $H_1 \subset \text{Int}\, H_0$ *and*

$$H_1 \cup (S^3 - \text{Int}\, H_0) \subset E_+^3 - E^2 \subset E^3 \cup \{\infty\} = S^3.$$

Let $h: S^3 \to S^3$ *be an orientation-preserving homeomorphism such that* $h(H_0) = H_1$. *Let* $i \ge 1$ *and* $n \ge 2$. *Then the set*

$$Y_i = Sp(h^{i+1}(H_0) \cup h^{-i-1}(S^3 - H_1)) \subset E^{n+2} \cup \{\infty\} = S^{n+2}$$

is not I-*equivalent to any set* $Y \subset S^{n+2}$ *with* $\pi_1(S^{n+2} - Y)$ *a free group.*

Proof. Let $G(i) = \pi_1(S^3 - h^{i+1}(H_0) - h^{-i-1}(S^3 - H_1))$. Then, from the definition of "spinning", $\pi_1(S^{n+2} - Y_i) \cong G(i)$. The proof of Theorem 1 showed that, for fixed i, there is a k with $G(i)/G(i)_k$ not isomorphic to F/F_k for any free group F. (F would necessarily have rank four.) The claim thus follows from Stallings' Proposition.

The four-dimensional case of the next theorem is included in Theorem 1.

THEOREM 3. *For each* $n \geq 5$ *there exists an arc* A *topologically embedded in the interior of a certain* PL n-*manifold* M^n, *such that no neighborhood of* A *in* M^n *is topologically embeddable in* S^n.

Proof. The proof follows closely the four-dimensional case. We give only a sketch. Choose h, H_0, and H_1 as in Lemma 4.3 and define the continua X_+, X_- in ${E_+}^3 - E^2 \subset S^3$ as before. Let A_+, A_- be disjoint, PL arcs in ${E_+}^3$ such that: A_+ joins X_+ to E^2, and Int A_+ misses $E^2 \cup X_+ \cup X_-$; and A_- joins X_- to E^2, and Int A_- misses $E^2 \cup X_+ \cup X_-$. Spin ${E_+}^3$ about E^2 by S^{n-4} to obtain $E^{n-1} \subset E^{n-1} \cup \{\infty\} = S^{n-1}$. Then, by Lemma 4.2, $Y_+ = Sp(X_+ \cup A_+)$ is strongly acyclic, and $Y_- = Sp(X_- \cup A_-)$ is cell-like. It is also easily verified that, since $\pi_1(E^3 - X_+) = \{1\}$, $E^{n-1} - Y_+ \approx Sp({E_+}^3 - X_+ - A_+)$ is simply connected. This is used to show that $\sum Y_+$ is PL cellular in M^n below.

We proceed as in Section 2, with Y_+, Y_- in $S^{n-1} = \partial B^n$ now playing the roles of X_+, X_-, respectively. We paste onto B^n two more disjoint copies of B^n along appropriate neighborhoods of Y_+, Y_- in S^{n-1} via the "identity" mapping to obtain M^n. We define X to be $\sum Y_- \cup \sum Y_+$. The reason that no neighborhood of X in M^n embeds in S^n is that (by Lemma 4.3) no neighborhood of

$$\sum Sp(X_+) \cup \sum Sp(X_-) \subset X$$

in M^n embeds in S^n. The corollary to the Edwards-Miller-Pixley-Eaton theorem (plus [2]) is used to shrink X down to an arc. Further details are left to the reader.

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