ON BANACH SPACES WITH UNIQUE ISOMETRIC PREDUALS

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A Banach space Y is called an isometric predual, or simply a predual, of a Banach space X if the dual Y^* of Y is isometrically isomorphic to X. A Banach space X is said to have a unique isometric predual if X has a predual and all preduals are mutually isometrically isomorphic. A. Grothendieck [5] first noticed the uniqueness of isometric preduals of L^∞-spaces and later S. Sakai generalized this to von Neumann algebras, see [7]. In this note a further generalization of Sakai's result will be proved. Evidently, infinite dimensional von Neumann algebras are the only previously known class of non-reflexive Banach spaces with unique isometric preduals; our generalization supplies a slightly broader class of such Banach spaces.

First we state one known lemma which is essentially due to Dixmier [2].

LEMMA 1. *Let* X *be a Banach space and let* X^* *be its dual.*

(a) *Suppose* Z *is a closed subspace of* X^* *satisfying the following two conditions;*

(i) Z *is total over* X *and minimal with respect to the property of being norm closed and total over* X,

(ii) Z *norms* X, *that is* $\|x\| = \sup\{\phi(x): \phi \in Z, \|\phi\| \leq 1\}$ *for all* $x \in X$.

Then X *is the dual of* Z *in the canonical way. Conversely any predual* Y *of* X *is isometrically isomorphic to a closed subspace* Z *of* X^* *having* (i) *and* (ii) *above.*

(b) *Statements* (i) *and* (ii) *above are together equivalent to*

(iii) *the closed unit ball of* X *is compact with respect to the weak topology on* X *induced by* Z.

A proof of (a) can be given using the Hahn-Banach theorem, and (b) is a direct consequence of Alaoglu's theorem and the bipolar theorem.

By definition, a von Neumann algebra X acting on a Hilbert space H is an algebra consisting of bounded operators on H, invariant under the adjoint operation, containing the identity operator 1_H on H and closed in the weak operator topology on the space of all bounded operators on H. The σ-weak operator topology on X is defined by the family of semi-norms $p(x) = \sum_{n=1}^{\infty} |\langle x\xi_n, \eta_n \rangle|$ $(x \in X)$, where $\{\xi_n\}$ and $\{\eta_n\}$ are sequences in H with $\sum_{n=1}^{\infty} \|\xi_n\|^2, \sum_{n=1}^{\infty} \|\eta_n\|^2 < +\infty$. A linear functional ϕ on X is called *normal* if ϕ is continuous in the σ-weak operator topology. The space of all normal linear functionals on X is denoted by X_*. It is well known that X_* is a closed subspace of the dual X^* of X, X is the dual of X_* in the canonical way and the σ-weak operator topology on X is equal to the weak-* topology on X as the dual of X_*, see [4].

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We will use, in an essential way, the following characterization of normal linear functionals which is due to M. Takesaki [8].

LEMMA 2. *A bounded linear functional* ϕ *on a von Neumann algebra* X *acting on a Hilbert space* H *is normal if and only if* ϕ *is completely additive, that is, we have* $\sum_{\lambda \in \Lambda} \phi(p_\lambda) = \phi(1_H)$ *for any orthogonal family of projections* p_λ $(\lambda \in \Lambda)$ *in* X *such that* $\sum_{\lambda \in \Lambda} p_\lambda = 1_H$.

Convergence of $\sum_{\lambda \in \Lambda} p_\lambda$ is as usual in the weak operator topology. This characterization was proved in [3] for positive linear functionals ϕ and later proved in [8] for general ϕ.

Now we state our result.

THEOREM. *Every quotient space of a von Neumann algebra by a* σ*-weakly closed subspace, as a Banach space with the quotient norm, has a unique isometric predual.*

Proof. Let X be a von Neumann algebra on a Hilbert space H and let X_* be the isometric predual of X consisting of all normal linear functionals on X. Suppose A is a σ-weakly closed subspace X. Then the annihilator $A^\perp$ of A in the dual X^* can be regarded as the dual of the quotient space X/A. The quotient space X/A can be regarded as the dual of $A^\perp \cap X_*$ in the canonical way, because the σ-weak operator topology on X is equal to the weak-* topology on X as the dual of X_*. Thus, with the above identifications, $A^\perp \cap X_*$ satisfies the two conditions (i) and (ii) of lemma 1, where "X" is replaced by X/A and "X^*" by $A^\perp$. In the following arguments we will show that $A^\perp \cap X_*$ is the only such closed subspace of $A^\perp$. If this is done, then lemma 1 gives the uniqueness of isometric preduals of X/A.

Now, let Z be a closed subspace of $A^\perp$ satisfying the two conditions (i) and (ii) of lemma 1. We claim that every element of Z is a normal linear functional on X, namely $Z \subset X_*$. Let p_λ $(\lambda \in \Lambda)$ be any orthogonal family of projections in X such that $\sum_{\lambda \in \Lambda} p_\lambda = 1_H$. Then for any finite subset E of Λ and any set of unimodular complex numbers δ_λ $(\lambda \in E)$ we have $\|\sum_{\lambda \in E} \delta_\lambda p_\lambda\| \leq 1$. Hence we have for all $\phi \in X^*$,

$$\sum_{\lambda \in \Lambda} |\phi(p_\lambda)| \leq \|\phi\| .$$

Then the well-defined infinite sum $\sum_{\lambda \in \Lambda} \psi(p_\lambda)$ for $\psi \in Z$ gives a bounded linear functional on Z. Thus, using (a) of lemma 1, there is an element of X/A, and hence an element q in X such that

$$\sum_{\lambda \in \Lambda} \psi(p_\lambda) = \psi(q) \qquad \text{for all } \psi \in Z . \tag{1}$$

For any finite subsets E and F of Λ with $E \subset F$ and any x in X with $\|x\| \leq 1$ we have $\|p_E x p_E + p_F - p_E\| \leq 1$, where $p_E = \sum_{\lambda \in E} p_\lambda$ and $p_F = \sum_{\lambda \in F} p_\lambda$. Therefore we have for all $\psi \in Z$ with $\|\psi\| \leq 1$,

$$|\psi(p_E x p_E)| + |\psi(p_F) - \psi(p_E)| \leq 1 .$$

Letting F approach Λ and using (1), we have for all $\psi \in Z$ with $\|\psi\| \leq 1$,

$$|\psi(p_E x p_E)| + |\psi(q) - \psi(p_E)| \leq 1.$$

Since the unit ball of Z is dense in the unit ball of $A^\perp$ in the weak-* topology on X^*, we have for all $\phi \in A^\perp$ with $\|\phi\| \leq 1$, $|\phi(p_E x p_E)| + |\phi(q) - \phi(p_E)| \leq 1$. In particular, this inequality holds for all $\phi \in A^\perp \cap X_*$ with $\|\phi\| \leq 1$. Now let E approach Λ. Since $p_E x p_E$ converges to x in the σ-weak operator topology on X, we have for all $\phi \in A^\perp \cap X_*$ with $\|\phi\| \leq 1$, $|\phi(x)| + |\phi(q) - \phi(1_H)| \leq 1$. Taking the supremum over all x with $\|x\| \leq 1$ in X, we conclude that $\|\phi\| + |\phi(q) - \phi(1_H)| \leq 1$ for all $\phi \in A^\perp \cap X_*$ with $\|\phi\| \leq 1$, so $\phi(q) - \phi(1_H) = 0$ for all $\phi \in A^\perp \cap X_*$. Thus q and 1_H represent the same element of X/A, and consequently we have

$$\phi(q) = \phi(1_H) \quad \text{for all } \phi \in A^\perp. \tag{2}$$

Combining (2) with (1), we have $\sum_{\lambda \in \Lambda} \psi(p_\lambda) = \psi(1_H)$ for all $\psi \in Z$. By lemma 2, this implies that every $\psi \in Z$ is normal, so $Z \subset X_*$. Therefore we have $Z \subset A^\perp \cap X_*$. Since $A^\perp \cap X_*$ is minimal with respect to the property of being norm closed and total over X/A, we must have $Z = A^\perp \cap X_*$. This completes the proof.

The following is clearly a direct consequence of our theorem.

COROLLARY. *Every closed subspace* A *of* $L^1(\Omega, \mu)$, *where* (Ω, μ) *is any measure space, is the unique isometric predual of the dual* A^* *of* A.

Finally we would like to give some remarks related to our result.

Remark 1. A definition of strong uniqueness of isometric preduals can be made; a Banach space X has a unique isometric predual *in the strong sense* if there is only one closed subspace Z in the dual X^* which satisfies (i) and (ii) of lemma 1 or equivalently (iii) of lemma 1. The proof of our theorem shows actually that every quotient space of a von Neumann algebra by a σ-weakly closed subspace has this strong uniqueness of isometric preduals. The author does not know any examples of Banach spaces which have uniqueness of isometric preduals but not in the strong sense.

Remark 2. In the introduction we mentioned that our theorem gives a slightly broader class of non-reflexive Banach spaces with unique isometric preduals than the class of infinite dimensional von Neumann algebras. We would like to justify this statement by giving two examples. The first example is the quotient space L^∞/H^∞, where L^∞ and H^∞ are the usual L^∞-space and the Hardy space on the unit circle. By our theorem this space has a unique isometric predual (namely, H^1), but L^∞/H^∞ is not isometrically isomorphic to any von Neumann algebra. Because if there were an isometric isomorphism, then, using the fact that the isometric predual of a von Neumann algebra is complemented in its second dual, see [7], we could prove that H^1 is complemented in L^1, which is a contradiction.

J. Lindenstrauss [6] showed that there is a closed subspace A of ℓ^1 such that A is not complemented in the second dual A^{**} of A. Our second example is the space A^*. By our corollary A is the unique isometric predual of A^*, and A^* is not isometrically isomorphic to any von Neumann algebra, since A is not complemented in A^{**}.

Remark 3. Recently T. Ando [1] proved that H^∞ has a unique isometric predual. It would be interesting to know whether more generally, any σ-weakly closed subalgebra of a von Neumann algebra has a unique isometric predual.

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