EVERY CRUMPLED n-CUBE IS A CLOSED n-CELL-COMPLEMENT

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Often it is a convenient simplification, in studying the wildness of (n - 1)-spheres topologically embedded in the n-sphere S^n, to suppose that the wildness is confined to one complementary domain, in that the closure of the other complementary domain is an n-cell. The principal result here, Theorem 6.1, shows that for $n \geq 5$ this simplification has validity in a stronger setting: for each crumpled n-cube C in S^n and $\varepsilon > 0$ there exists an ε-homeomorphism h of C into S^n such that the closure of S^n - h(C) is an n-cell. For n = 3 the same result has been established by Hosay [16] and Lininger [17].

A *crumpled* n-*cube* C is a space homeomorphic to the union of an (n - 1)-sphere in S^n and one of its complementary domains; the subset of C consisting of those points at which C is an n-manifold (without boundary) is called the *interior of* C, written Int C, and the subset C - Int C, which corresponds to the given (n - 1)-sphere, is called the *boundary of* C, written Bd C. A crumpled n-cube C is a *closed* n-*cell-complement* if there exists an embedding h of C in S^n such that S^n - h(Int C) (equivalently, the closure of S^n - h(C)) is an n-cell. Translated into this terminology, the principal result implies that for $n \geq 5$ each crumpled n-cube is a closed n-cell-complement (Corollary 6.4).

Besides validating this simplification, the paper supports the opposite process permitting the construction of complexities. To describe the construction, we look first at a standard situation: any (n - 1)-sphere Σ in S^n bounds two crumpled n-cubes C_0 and C_1, and fastening C_0 and C_1 (abstractly conceived) back together along their boundaries in an appropriate way reproduces S^n, with both Bd C_0 and Bd C_1 identified as Σ. Generally, such an attaching is called a sewing; specifically, a *sewing* of two crumpled n-cubes C_0 and C_1 is a homeomorphism between their boundaries, and associated with a sewing h is the *sewing space*, denoted as $C_0 \cup_h C_1$, which is the quotient space obtained from the disjoint union of C_0 and C_1 under identification of each point x in Bd C_0 with its image h(x) in Bd C_1. To construct (n - 1)-spheres in S^n with wildness in both complementary domains, one first can select crumpled n-cubes C_0 and C_1, together with a sewing h, and then can hope to prove that $C_0 \cup_h C_1$ is homeomorphic to S^n. The ultimate problem, determining whether a given sewing space $C_0 \cup_h C_1$ is topologically equivalent to S^n, can be an intricate and complex puzzle, about which [14] supplies much information. In any event, the results here (see Corollary 6.7) imply that the sewing space $C_0 \cup_h C_1$ is homeomorphic to a decomposition space associated with a decomposition of S^n into points and flat arcs, thereby reducing the sewing problem to a decomposition problem.

Instrumental for the approach used here is Ancel and Cannon's recent solution [1] of the Locally Flat Approximation Theorem, which states that each embedding of an (n - 1)-manifold in an n-manifold ($n \geq 5$) can be approximated arbitrarily closely by locally flat embeddings. Earlier Bryant, Edwards and Seebeck [5] had devised a

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significant attack on this theorem, based on techniques of Štan'ko [23], and Štan'ko himself produced a comparable attack [24], both of which essentially manipulated one side at a time and in effect attempted to reembed crumpled n-cubes as closed n-cell-complements. In both cases the argument is satisfactory for reembedding a large class of crumpled n-cubes (see [14, Section 6] for further discussion of the Bryant-Edwards-Seebeck results) but is incomplete. The work of Ancel and Cannon proceeds with similar techniques, but rather than reembedding crumpled cubes, they cleverly embed decompositions of which the crumpled cubes form the decomposition spaces.

Organizing this discussion around crumpled n-cubes and embeddings of (n - 1)-spheres serves as an advantageous shorthand for us. Nevertheless, because the arguments are characteristically local, they can be adapted to a wider setting concerning embeddings of (n - 1)-manifolds in n-manifolds, illustrated by the following analogue to the main theorem:

THEOREM 6.1'. *Suppose that* N *is an* n-*manifold* (*without boundary*), $n \geq 5$, C *is a closed subset of* N *such that the boundary of* C (*relative to* N) *is an* (n - 1)-*manifold, and* ε: C $\to$ (0, 1) *is continuous. Then there exists an embedding* h *of* C *in* N *such that* $\rho(c, h(c)) < \varepsilon(c)$ *for each point* c *in* C *and* h(Bd C) *is collared from* N - h(C); *that is, there is an embedding* H *of* Bd C $\times$ [0, 1] *in* $\mathscr{C}\ell$(N - h(C)) *such that* $H(\langle b, 0\rangle) = h(b)$ *for each* $b \in$ Bd C.

A word about the structure of this paper: the main theorem depends heavily upon a weak version of itself that applies to a restricted class of crumpled n-cubes. The various types of crumpled cubes are discussed in Section 2. A criterion for detecting the restricted type, which is similar to criteria in [13] but with the dimension restrictions improved, and which may serve functions outside this paper, is given in Section 3. Technical results providing controls that force approximations to (n - 1)-spheres to bound crumpled n-cubes of this type are given in Section 4. Another highly technical result, describing controls allowable in the shrinking of certain arcs and leading to a decomposition theorem pertaining to a countable collection of decompositions for which the underlying point sets of the nondegenerate elements form a null sequence of pairwise disjoint sets, is stated in Section 5. Finally, building on all of these, the proof of the main theorem is given in Section 6.

A reasonable method for studying this paper is to read Section 2, the statements of Corollary 3.4 and Lemma 4.3, and Section 5 and then to concentrate on Section 6, putting aside the technical details of Sections 3 and 4.

1. DEFINITIONS AND NOTATION

The symbol ρ is used throughout to denote a fixed metric on S^n. For maps f and g of a space X into S^n, $\rho(f, g)$ denotes the least upper bound of

$$\{\rho(f(x), g(x)): x \in X\}.$$

The symbol 1 is often employed to denote, somewhat ambiguously but usually rather clearly in context, the identity map and, for $A \subset X$, $1|A$ is used to denote the inclusion of A in X. Thus, for $A \subset S^n$ and $\varepsilon > 0$, an embedding h of A in S^n is an ε-*homeomorphism* if and only if $\rho(h, 1|A) < \varepsilon$.

The symbol Δ^k denotes a canonical k-simplex and $\partial\Delta^k$ its boundary. Let $X \subset Y \subset Z$, where Z is a metric space, and $k \geq 0$ an integer. Then X is *locally* k-*connected in* Y *at a point* $y \in \mathscr{C}\ell\, Y$ ("$\mathscr{C}\ell$" denotes closure), written X is k - LC in Y at y, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any map of $\partial\Delta^{k+1}$ into $X \cap N_\delta(y)$ can be extended to a map of Δ^{k+1} into $Y \cap N_\varepsilon(y)$; and X *is uniformly* k-*connected in* Y, written X is k-ULC in Y, if a $\delta > 0$ exists that is independent of the choice of $y \in \mathscr{C}\ell\, Y$. Generally, one abbreviates the phrase "Y is k-ULC in Y" to "Y is k-ULC." For our purposes the standard choices of k occur for k = 0 or k = 1, and the triple (X, Y, Z) occurs with Z representing a crumpled n-cube, X representing its interior, and Y representing simply some set for which $X \subset Y \subset Z$, in which case Y is 1-ULC if and only if X is 1-ULC in Y [7, Theorem 2C.5]. Section 2 of Cannon's paper [7] is a comprehensive source of related information about the LC properties.

2. DEMENSION THEORY AND THE TYPES OF CRUMPLED CUBES

As a crude measure of the (one-sided) wildness of the boundary sphere, it is beneficial to distinguish, as in [14], certain types of crumpled n-cubes C by enumerating properties that allow increasing complexity of that wildness, as follows:

Type 1. There exists a 0-dimensional F_σ set F in Bd C such that F is a countable union of Cantor sets that are tame relative to Bd C and F ∪ Int C is 1-ULC. The only subtype of any consequence for this paper, its simplicity has afforded previous solutions to the main problem for crumpled cubes of this type.

Type 2. There exists a 0-dimensional F_σ set F in Bd C such that F ∪ Int C is 1-ULC.

Type 3. There exists a 1-dimensional F_σ set F in Bd C such that F ∪ Int C is 1-ULC.

Bing [3] has shown that each crumpled 3-cube is of Type 1 (in case n = 3, Types 1 and 2 are indistinguishable). Daverman [10] has shown that each crumpled n-cube is of Type 3 in case $n \geq 5$ and Ancel and McMillan [2] have proved the same result in case n = 4. It is suspected that examples of Type 3 crumpled n-cubes exist, although none has been discovered; examples of Type 2 are known (see [12, Theorem 5.4] and [14, Section 13]).

The theory of demension, or embedding dimension, as developed by Štan'ko [21, 22] for compacta and extended by Edwards [15] to σ-compacta, supplies functional terminology for describing and distinguishing the types of crumpled cubes. We shall review the significant features of the theory to be employed in this paper.

Suppose G is an open subset of a PL q-manifold Q and $\varepsilon > 0$. Suppose there is a compact k-dimensional subpolyhedron P of Q contained in G, there is a compact metric space Y, and there is a continuous proper surjection $s: Y \times [0, 1) \to G$ such that $s^{-1}(P) = Y \times \{0\}$, $S|Y \times (0, 1)$ is injective, and $\operatorname{diam} s(y \times [0, 1)) < \varepsilon$ for each y in Y. In this situation we say that G is an *open* ε-*mapping cylinder neighborhood of* P *in* Q and that P is a k-*spine of* G. When Q is compact and boundaryless, open ε-mapping cylinder neighborhoods naturally arise as the complements of (q - k - 1)-skeletons in triangulations of Q having mesh less than ε.

Suppose now that X is a nonempty compact subset of the interior of Q. For an integer $k \geq 0$ we say that the *embedding dimension of* X *in* Q *is at most* k, abbreviated as $\operatorname{dem}_Q X \leq k$, if for each $\varepsilon > 0$, X is contained in an open ε-mapping cylinder neighborhood with a k-spine, and we say that the *embedding dimension* (or the *demension*) *of* X *in* Q *is* k, abbreviated as $\operatorname{dem}_Q X = k$, if $\operatorname{dem}_Q X \leq k$ but not $\operatorname{dem}_Q X \leq k - 1$. (For technical reasons set $\operatorname{dem}_Q \phi = -\infty$.) For a nonempty σ-compact subset F of Q we define the *embedding dimension* of F *in* Q, written $\operatorname{dem}_Q F$, as $\max\{\operatorname{dem}_Q X: X \text{ is a compact subset of } F\}$.

In case X is a Cantor set in the interior of Q, then X is tame in the usual sense if and only if $\operatorname{dem}_Q X = 0$ if and only if $\operatorname{dem}_Q X \leq q - 3$ (see [21, 15]; in particular, this holds even when $q = 4$ [15, p. 206]). Accordingly, a crumpled n-cube C is of Type 1 if and only if there exists a σ-compact set F in $\operatorname{Bd} C$ such that $\operatorname{dem}_{\operatorname{Bd} C} F = 0$ and $F \cup \operatorname{Int} C$ is 1-ULC. Here it is frequently helpful to know that if $X_1, X_2, \cdots$ are compacta in $\operatorname{Bd} C$ for which $\operatorname{dem}_{\operatorname{Bd} C} X_i \leq k$ then

$$\operatorname{dem}_{\operatorname{Bd} C} \cup X_i \leq 0$$

[15, Prop. 1.1].

It is shown in [14] that the crumpled n-cubes C $(n \geq 5)$ of Type 2 can be separated into two classes, those for which $C \cup_{Id} C = S^n$ (where Id denotes the identity sewing) and those for which $C \cup_{Id} C \neq S^n$. For definiteness, we say that the crumpled n-cubes in the first class are of Type 2A and those in the second class are of Type 2B. Suppose now that the crumpled n-cube C is embedded in S^n so that $S^n - \operatorname{Int} C$ is an n-cell. It follows from Theorem 10.1 [14] that C is of Type 2A if and only if $\operatorname{Bd} C$ contains a 0-dimensional F_σ set F such that $\operatorname{dem}_{S^n} F = 0$ and $F \cup \operatorname{Int} C$ is 1-ULC. Thus, the distinction between crumpled cubes of Type 2B and Type 3 are found in *dimension* theory but not in *demension* theory (because in either case, if $F \subset \operatorname{Bd} C$ satisfies $F \cup \operatorname{Int} C$ is 1-ULC, then $\operatorname{dem}_{\operatorname{Bd} C} F \geq n - 3$ and $\operatorname{dem}_{S^n} F \geq n - 2$), while the distinctions among Types 1, 2A and 2B are found in *demension* theory but not (entirely) in *dimension* theory. Such limitations on the applicability of demension theory to this context appear intrinsically connected with the difficulties encountered in seeking Type 3 crumpled n-cubes.

3. IDENTIFICATION OF CRUMPLED CUBES OF TYPE 1

As suggested in the introduction, the information from this section vital for later applications is set forth in Corollary 3.4, a reproduction of a result due to Ancel and McMillan [2, Theorem 2], slightly reformulated and given with a decreased dimension restriction. Their result depends on the criteria of [13] for identifying Type 1 crumpled n-cubes $(n \geq 6)$ and Theorem 3.3 here improves a portion of those criteria by lowering the dimension restriction to $n \geq 5$.

The fundamental technical result, comparable to Lemma 1 of [13], is given by Lemma 3.1 below. It should be emphasized that a PL triangulation of a sphere Σ in S^n endows Σ with a structure in no way presumed to be consistent with preassigned structures on S^n or Σ. It should also be admitted that we abuse notation somewhat by allowing $R^{(2)}$ to stand for the underlying point set of the 2-skeleton of a triangulation R.

LEMMA 3.1. *Suppose Σ is an $(n - 1)$-sphere in S^n $(n \geq 5)$, W is a component of $S^n - \Sigma$ such that there exist PL triangulations R of Σ of arbitrarily small mesh for which $\mathscr{C}\ell\, W - R^{(2)}$ is 1-ULC, $f: \Delta^2 \to \mathscr{C}\ell\, W$ is a map such that $f(\partial\Delta^2) \subset W$, and $\varepsilon > 0$. Then there exists a map $f': \Delta^2 \to \mathscr{C}\ell\, W$ and there exists a PL triangulation T of Σ such that*

(i) $f'|\partial\Delta^2 = f|\partial\Delta^2$,

(ii) $\rho(f', f) < \varepsilon$,

(iii) $\operatorname{mesh} T < \varepsilon$,

(iv) $f'(\Delta^2) \cap T^{(2)} = \emptyset$, and

(v) the diameter of each component of $f'(\Delta^2) \cap \Sigma$ is less than ε.

Proof. By [7, Corollary 2C.2.1] we can assume that $f^{-1}(f(\Delta^2) \cap \Sigma)$ is 0-dimensional.

Step 1. Determine $\delta > 0$ such that any δ-subset of Σ is contained in an open $(n-1)$-cell of diameter less than $\varepsilon/10$. Cover $f^{-1}(\Sigma)$ by a finite collection of pairwise disjoint open 2-cells $Y_1, \cdots, Y_k$ in Int Δ^2 so small that diam $f(Y_i) < \delta$. We shall obtain an approximation to f coinciding with f off $\bigcup Y_i$ such that the images of the Y_i's are pairwise disjoint. To begin, approximate f by a map $s: \Delta^2 \to S^n$ such that

(1) $s|\Delta^2 - \bigcup Y_i = f|\Delta^2 - \bigcup Y_i$,

(2) $\rho(s, f) < \varepsilon/10$,

(3) diam $s(Y_i) < \delta$,

(4) $s(Y_i) \cap s(Y_j) = \emptyset$, for $i \neq j$.

Now to employ the operative hypothesis, choose $\alpha > 0$ such that

(5) $\alpha < (\rho(s(Y_i) \cap \Sigma, s(Y_j) \cap \Sigma))/3$, $i \neq j$,

and then name a triangulation T of Σ such that

(6) mesh $T < \min\{\alpha, \varepsilon/10\}$,

(7) $\mathscr{C}\ell\, W - T^{(2)}$ is 1-ULC.

Step 2. Let X denote the component of $\Delta^2 - s^{-1}(\Sigma)$ containing $\partial\Delta^2$. Apply the Tietze Extension Theorem to cut $s(Y_i)$ off on an $(\varepsilon/10)$-cell in Σ, obtaining a map $t: \Delta^2 \to \mathscr{C}\ell\, W$ such that

(8) $t|X = s|X$ (implying $t|\Delta^2 - \bigcup Y_i = f|\Delta^2 - \bigcup Y_i$),

(9) diam $t(Y_i) < 2\varepsilon/10$,

(10) $\rho(t, f) < 3\varepsilon/10$,

(11) $t^{-1}(\Sigma) = \Delta^2 - X$.

To improve the situation as much as possible, we can assume that $t|\Delta^2 - \mathscr{C}\ell\, X$ is locally piecewise linear as a map into Σ and in general position with respect to T so that $(\Delta^2 - \mathscr{C}\ell\, X) \cap t^{-1}(T^{(2)})$ is a locally finite subset of $\Delta^2 - \mathscr{C}\ell\, X$.

Step 3. For $i = 1, \cdots, k$ let $R_i = \bigcup_{j \neq i} \mathrm{St}(s(Y_j) \cap \Sigma, T)$ and let

$$Q_i = \mathrm{St}(s(Y_i) \cap \Sigma, T).$$

Restrictions on α and T were structured so that

(12) $Q_i \cap R_i = \emptyset$ and $Q_i \cap Q_j = \emptyset$, for $i \neq j$.

For $i = 1, \cdots, k$ there exists a compact 2-manifold (with boundary) H_i such that

(13) $Y_i \cap \mathrm{Fr} X \subset \mathrm{Int}\, H_i \subset H_i \subset Y_i$,

(14) $t(\partial H_i) \cap T^{(2)} = \emptyset$,

(15) $t(H_i) \cap \Sigma \subset \mathrm{Int}\, Q_i$.

Approximate t by a map u: $\Delta^2 \to \mathscr{C}\ell\, W$ to send $\bigcup H_i$ into the compliment of $T^{(2)}$. Specifically, u satisfies

(16) $\rho(u, f) < 4\varepsilon/10$,

(17) diam $u(Y_i) < 4\varepsilon/10$,

(18) $u|\Delta^2 - \bigcup H_i = t|\Delta^2 - \bigcup H_i$,

(19) $u(H_i) \cap \Sigma \subset \text{Int } Q_i - T^{(2)}$.

Now define B_i as $Y_i - (H_i \cup X)$. In the first derived subdivision T' of T, let T^* denote the subcomplex dual to $T^{(2)}$, that is, $T^* = \{\sigma \in T' : \sigma \cap T^{(2)}\} = \emptyset$. After another slight general position adjustment to $u|\bigcup B_i$ such that $u(B_i) \cap T^* = \emptyset$, we deform $u(B_i) \cap R_i$ across the join structure of T' into $T^{(2)} \bigcup \text{Bd } R_i$, defining a map v: $\Delta^2 \to \mathscr{C}\ell\, W$ such that

(20) $v|\Delta^2 - \bigcup B_i = u|\Delta^2 - \bigcup B_i$,

(21) $v\left(\bigcup B_i\right) \subset \Sigma$,

(22) $\rho(v, u) < \text{mesh } T < \varepsilon/10$ (implying $\rho(v, f) < 5\varepsilon/10$),

(23) diam $v(Y_i) < 6\varepsilon/10$,

(24) $v(B_i) \cap \text{Int } R_i \subset T^{(2)}$.

Step 4. Conditions (24), (19) and (20) imply that $v(\mathscr{C}\ell\, B_i) \cap v(H_j) = \emptyset$, $i \neq j$, and conditions (12), (19) and (20) imply that $v(H_i) \cap v(H_j) = \emptyset$, $i \neq j$. Consequently, the remaining concern is to make the images of the B_i's be disjoint.

Without changing v off $\bigcup B_i$, we can assume again that $v|\bigcup B_i$ is locally piecewise linear as a map into Σ and is in general position. Since dim ≥ 4 and since $v(\text{Bd } B_i) \cap v(\mathscr{C}\ell\, B_j) = \emptyset$ for $i < j$, $v(B_i) \cap v(B_j)$ is a finite set, which we enumerate as $v(B_i) \cap v(B_j) = \{p_{ijk} : k = 1, \cdots, m(i, j);\ i < j\}$. Furthermore, there exists a mutually exclusive family $\{N_{ijk}\}$ (indices ranging over all possible i, j, k) of compact, PL neighborhoods of P_{ijk}, respectively, intersecting $\bigcup_m v(B_m)$ only in $v(B_i) \cup v(B_j)$, for which there exist homeomorphisms of N_{ijk} into (n - 1)-space carrying $N_{ijk} \cap v(B_i)$ and $N_{ijk} \cap v(B_j)$ into 2-dimensional hyperplanes. Since we can require that $v\left(\Delta^2 - \bigcup B_i\right) \cap \Sigma$ separates no open set in Σ (either routinely forcing $v\left(\Delta^2 - \bigcup B_i\right)$ to be 2-dimensional or using [10] to make

$$v\left(\Delta^2 - \bigcup B_i\right) \cap \Sigma$$

be 1-dimensional), there exists a homeomorphism Θ of Σ onto itself such that

(25) $\Theta|\Sigma \cap v\left(\Delta^2 - \bigcup B_i\right) = 1|\Sigma \cap v\left(\Delta^2 - \bigcup B_i\right)$,

(26) $\rho(\Theta, 1|\Sigma) < \text{mesh } T < \varepsilon/10$,

(27) $\Theta(N_{ijk}) \cap T^{(2)} = \Theta(N_{ijk} \cap v(B_i))$.

Now we define a map w of Δ^2 into $\mathscr{C}\ell\, W$ as

$$w|\Delta^2 - \bigcup B_i = v|\Delta^2 - \bigcup B_i \quad \text{and} \quad w|\bigcup B_i = \Theta v|\bigcup B_i.$$

Clearly,

(28) $\rho(w, v) < \varepsilon/10$ (implying $\rho(w, f) < 6\varepsilon/10$),

(29) $\operatorname{diam} w(Y_i) < 8\varepsilon/10$.

Since $\mathscr{C}\ell\, W - T^{(2)}$ is 1-ULC, there exists a map g of Δ^2 into $\mathscr{C}\ell\, W$ such that

(30) $g|\Delta^2 - \bigcup B_i = w|\Delta^2 - \bigcup B_i$

(31) $\rho(g, w) < \varepsilon/10$ (implying $\rho(g, f) < \varepsilon$),

(32) $\operatorname{diam} g(Y_i) < \varepsilon$,

(33) $g|B_j - w^{-1}\left(\bigcup\{N_{ijk}: i < j\}\right) = w|B_j - w^{-1}\left(\bigcup\{N_{ijk}: i < j\}\right)$,

(34) $g(B_j \cap w^{-1}(N_{ijk})) \subset \Theta(N_{ijk}) - T^{(2)}, \quad i < j$.

The relevance of this rearrangement scheme is displayed by conditions (27), (33) and (34), which imply, for $i < j$, that

(35) $g(B_j \cap w^{-1}(N_{ijk})) \subset \Theta(N_{ijk}) - w(B_i) \subset \Sigma - g(B_i)$

and which combine with condition (25) and the definition of w to yield

(36) $g(B_j) \cap g(H_i) = g(B_j) \cap v(H_i) = \emptyset, \quad i \neq j$.

It follows then that the sets $g(Y_i) \cap \Sigma$ are pairwise disjoint sets of diameter less than ε.

Finally, again because $\mathscr{C}\ell\, W - T^{(2)}$ is 1-ULC, there exists a map

$$f': \Delta^2 \to \mathscr{C}\ell\, W - T^{(2)}$$

approximating g so closely that, as required,

(37) $\rho(f', f) < \varepsilon$,

(38) $f'|\Delta^2 - \bigcup Y_i = f|\Delta^2 - \bigcup Y_i$,

(39) $f'(Y_i) \cap f'(Y_j) \cap \Sigma = \emptyset, \quad i \neq j$,

(40) $\operatorname{diam} f'(Y_i) < \varepsilon$.

LEMMA 3.2. *Under the hypotheses of Lemma* 3.1, *there exists a map* g *of* Δ^2 *into* $\mathscr{C}\ell\, W$ *such that* $\rho(g, f) < \varepsilon$ *and* $\operatorname{dem}_\Sigma g(\Delta^2) \cap \Sigma \leq 0$.

Proof. The map g results as the limit of a sequence of maps g_i, obtained from Lemma 3.1, subject to controls that force $g(\Delta^2) \cap \Sigma$ to be 0-dimensional and such that for each positive integer i there exist triangulations R_i of Σ of mesh less than $1/i$ whose 2-skeleta do not intersect $g(\Delta^2)$. Thus, $\operatorname{dem}_\Sigma g(\Delta^2) \cap \Sigma \leq n - 4$, implying that $\operatorname{dem}_\Sigma g(\Delta^2) \cap \Sigma \leq 0$ [15, 21]. Further details of this and of the next theorem are given in [13]. The proof of the next theorem in case $n = 4$ is left for the reader to establish directly.

THEOREM 3.3. *Suppose* Σ *is an* $(n - 1)$*-sphere in* S^n $(n \geq 4)$ *and* W *is a component of* $S^n - \Sigma$. *Then there exist* PL *triangulations* R *of* Σ *of arbitrarily small mesh for which* $\mathscr{C}\ell\, W - R^{(2)}$ *is* 1-ULC *if and only if there exists a subset* F *of* Σ *such that* $\operatorname{dem}_\Sigma F = 0$ *and* $F \cup W$ *is* 1-ULC.

In contrast with [13, Theorem 4], unresolved is the question whether the conditions stated in Theorem 3.3 are equivalent to the existence of PL triangulations R of Σ having arbitrarily small mesh whose 2-skeleta are tame in S^n.

COROLLARY 3.4. *Suppose* C *is a crumpled* n-*cube* ($n \geq 5$) *with* Bd C = Σ, *and* F *is a* σ-*compactum in* Σ *such that* $\text{dem}_\Sigma F \leq n - 4$ *and* F $\cup$ Int C *is* 1-ULC. *Then* C *is of Type* 1; *that is, there exists a* σ-*compactum* F' *in* Σ *such that* $\text{dem}_\Sigma F' \leq 0$ *and* F' $\cup$ Int C *is* 1-ULC.

Proof. It suffices to obtain triangulations R of Σ of arbitrarily small mesh for which Int C is 1-ULC in C - $R^{(2)}$, and since Int C is 1-ULC in F $\cup$ Int C, it suffices to obtain triangulations R of Σ of arbitrary small mesh such that

$$R^{(2)} \cap F = \emptyset.$$

Fix $\varepsilon > 0$, and let T denote a triangulation of Σ of mesh less than $\varepsilon/3$; according to [15, Prop. 2.2], there exists an $(\varepsilon/3)$-homeomorphism H of Σ onto itself such that $H(T^{(2)}) \cap F = \emptyset$ and $H(T) = \{H(\sigma)\colon \sigma \in T\}$ is a triangulation of Σ of mesh less than ε whose 2-skeleton avoids F, as desired.

4. APPROXIMATIONS BOUNDING CRUMPLED CUBES OF TYPE 1

The results of this section culminate in Lemma 4.3, which for the proof of the main lemma here permits reduction of the original problem to a simpler case, by showing that any (n - 1)-sphere Σ in S^n contains Sierpiński curves X such that every (n - 1)-sphere Σ' containing X that is locally flat modulo X and that is a sufficiently close approximation to Σ bounds crumpled n-cubes of Type 1. The examples in [12] of crumpled n-cubes not of Type 1, bounded by spheres that are locally flat modulo Cantor sets, show why the Sierpiński curves X must be so carefully constructed.

LEMMA 4.1. *Let* F *be a* k-*dimensional* F_σ *set in* S^q, *where* $k \leq q - 3$ *and* $q \geq 4$. *Then* S^q *contains a countable collection* $\{C_i\}$ *of Cantor sets such that* $\text{dem}_{S^q} \bigcup C_i = 0$ *and each compactum* K *in* F - $\bigcup C_i$ *satisfies* $\text{dem}_{S^q} K \leq k$.

Proof. Choose a sequence of triangulations $\{T_m\}$ of S^q such that T_{m+1} is a subdivision of T_m and the mesh of T_m is less than $1/m$. Since $q \geq 4$, the 1-skeleta of all the T_m's can be isotopically adjusted so as to miss F, making the intersection of F with each $T_m^{(2)}$ be 0-dimensional. There exists a family $\{C_i\}$ of tame Cantor sets in S^q such that $\bigcup C_i$ contains $\bigcup (F \cap T_m^{(2)})$.

Any compactum K in F - $\bigcup C_i$ then is contained in $S^q - T_m^{(2)}$ for $m = 1, 2, \cdots$, implying that $\text{dem}_{S^q} K \leq q - 3$. Hence, $\text{dem}_{S^q} K \leq \dim K \leq k$.

LEMMA 4.2. *Let* Σ *denote an* (n - 1)-*sphere in* S^n, F *a subset of* Σ *such that each component* U *of* $S^n - \Sigma$ *is* 1-ULC *in* U $\cup$ F, *and* X *a compact subset of* Σ. *Then there exists a continuous function* $\delta\colon \Sigma \to [0, 1)$, *with* $X = \delta^{-1}(0)$, *such that for any embedding* e *of* Σ *in* S^n *satisfying*

(i) $\rho(z, e(z)) \leq \delta(z)$ *for each* $z \in \Sigma$,

(ii) $e(\Sigma)$ *is locally flat modulo* $e(X) = X$,

each component U^* *of* $S^n - e(\Sigma)$ *is* 1-ULC *in* $U^* \cup (F \cap X)$.

Proof. Let U_0 and U_1 denote the components of $S^n - \Sigma$. Choose points $p_0 \in U_0$ and $p_1 \in U_1$ and define W as $S^n - (X \cup \{p_0, p_1\})$. The map $\delta\colon \Sigma \to [0, 1)$ will be obtained so that any embedding e satisfying condition (i) will have

$e(\Sigma) \subset \Sigma \cup W$, among other properties. For any such embedding e, we shall let U_j^* denote the component of $S^n - e(\Sigma)$ containing $U_j - W$, $j = 0, 1$. It will suffice to describe how to limit the mapping δ so that U_0^* is 1-ULC in $U_0^* \cup (F \cap X)$.

There exists a neighborhood V of $\Sigma - X$, with $V \subset W$, such that one can construct a deformation $d_t: \Sigma \cup V \to \Sigma \cup W$ such that $d_0 = 1|\Sigma \cup V$, $d_t|\Sigma = 1|\Sigma$, $d_t(V) \subset W$, and d_1 is a retraction of V to $\Sigma - X$. Let g_t represent the deformation of $\mathscr{C}\ell\, U_0 \cup V$ in $\mathscr{C}\ell\, U_0 \cup W$ defined by

$$g_t|\mathscr{C}\ell\, U_0 = 1|\mathscr{C}\ell\, U_0 \quad \text{and} \quad g_t|V - U_0 = d_t|V - U_0 .$$

Fix a map $\delta: \Sigma \to [0, 1)$, with $X = \delta^{-1}(0)$, such that, whenever e is an embedding of Σ in S^n satisfying condition (i), then $e(\Sigma) \subset \Sigma \cup V$ and the "straight line" homotopy

$$\psi_t(z) = ((1 - t)z + te(z))/\|(1 - t)z + te(z)\|$$

between the inclusion and e yields $\psi_t(\Sigma - X) \subset V$. (Here S^n is regarded as a subset of E^{n+1}).

Now suppose that e is an embedding satisfying conditions (i) and (ii). We shall prove that U_0^* is 1-ULC in $U_0^* \cup (F \cap X)$. To that end, since $e(\Sigma)$ is locally flat at each point of $e(\Sigma - X)$, it will suffice to show that U_0^* is 1-LC in $U_0^* \cup (F \cap X)$ at each point of $e(X) = X$.

By variations to Borsuk's Homotopy Extension Theorem, similar to those of [20, pp. 31-32], there exists a deformation $D_t: \Sigma \cup V \to \Sigma \cup W$ such that $D_0 = 1|\Sigma \cup V$, $D_t|e(\Sigma) = 1|e(\Sigma)$, $D_t(V) \subset W$ and D_1 is a retraction of $\Sigma \cup V = e(\Sigma) \cup V$ onto $e(\Sigma)$. As before, define a deformation G_t of $\mathscr{C}\ell\, U_0^* \cup V$ into $\mathscr{C}\ell\, U_0^* \cup W$ as

$$G_t|\mathscr{C}\ell\, U_0^* = 1|\mathscr{C}\ell\, U_0 \quad \text{and} \quad G_t|V - U_0^* = D_t|V - U_0^* .$$

Consider any neighborhood N of a point x in X (it is most convenient for the rest of the argument to employ neighborhoods of x in $\mathscr{C}\ell\, U_0 \cup V = \mathscr{C}\ell\, U_0^* \cup V$). There exists a neighborhood N' of x such that $G_1(N') \subset N$, there exists a neighborhood $N'' \subset N'$ of x such that any loop in $N'' \cap U_0$ is contractible in $N' \cap (U_0 \cup F)$, and there exists another neighborhood $N''' \subset N''$ of x such that $g_t(N''') \subset N''$ for $t \in [0, 1]$. Consider an arbitrary loop $f: \partial\Delta^2 \to N''' \cap U_0^*$. Then g_t provides a homotopy in $N'' - X$ between f and a loop f' in $(N'' - X) \cap \mathscr{C}\ell\, U_0$, and f' is homotopic in $N'' - X$ to a loop in $(N'' - X) \cap U_0$, which in turn is contractible in $N' \cap (U_0 \cup F)$. These homotopies can be spliced to form a map $m: \Delta^2 \to N'$, extending f, such that $m(\Delta^2) \cap X \subset F$. Thus, $G_1 m(\Delta^2) \subset N \cap \mathscr{C}\ell\, U_0^*$, $G_1 m(\Delta^2) \cap X \subset F$, and $G_1 m|\partial\Delta^2 = f$. Since $e(\Sigma)$ is locally flat modulo X, we can adjust the map $G_1 m$ in a collar on $e(\Sigma - X)$ so that $G_1 m(\Delta^2) \subset N \cap (U_0 \cup (F \cap X))$ to complete the proof.

A $(q - 1)$-*dimensional Sierpiński curve* is a compact space X that can be embedded in S^q, say by an embedding h, such that the components of $S^q - h(X)$ form a null sequence $U_1, U_2, \cdots$ satisfying

(i) $S^q - U_i$ is a q-cell for each i,

(ii) $\mathscr{C}\ell\, U_i \cap \mathscr{C}\ell\, U_j = \emptyset$, $i \neq j$, and

(iii) the union of the U_i is dense in S^q.

Cannon has shown [6] that any two $(q - 1)$-dimensional Sierpiński curves are homeomorphic $(q \neq 4)$. We say that an embedding h of a $(q - 1)$-dimensional Sierpiński curve X in S^q is *standard* if the closure of each component of $S^q - h(X)$ is a q-cell.

LEMMA 4.3. *Let* Σ *denote an* $(n - 1)$*-sphere in* S^n $n \geq 5$, *and let* $\varepsilon > 0$. *There exists an* $(n - 2)$*-dimensional Sierpiński curve* X *in* Σ *such that* X *is standardly embedded in* Σ *and each component of* $\Sigma - X$ *has diameter less than* ε, *and there exists a continuous function* $\delta: \Sigma \to [0, 1)$, *with* $X = \delta^{-1}(0)$, *such that for any embedding* e *of* Σ *in* S^n *satisfying*

(i) $\rho(z, e(z)) \leq \delta(z)$ *for each* $z \in \Sigma$,

(ii) $e(\Sigma)$ *is locally flat modulo* $e(X) = X$,

the sphere $e(\Sigma)$ *bounds two crumpled* n*-cubes of Type* 1.

Proof. By [10, Theorem 2] Σ contains a 1-dimensional F_σ set F such that each component U of $S^n - \Sigma$ is 1-ULC in $U \cup F$. By Lemma 4.1 Σ contains a countable collection $\{C_i\}$ of Cantor sets such that $\text{dem}_\Sigma \bigcup C_i = 0$ and each compactum K in $F - \bigcup C_i$ satisfies $\text{dem}_\Sigma K \leq 1$. Because any $(n - 2)$-dimensional Sierpiński curve in Σ can be isotopically (relative to Σ) pushed off $\bigcup C_i$, Σ contains an $(n - 2)$-dimensional curve X such that X is standardly embedded in Σ, each component of $\Sigma - X$ has diameter less than ε, and $X \cap \left(\bigcup C_i\right) = \emptyset$. One applies Lemma 4.2 to obtain the required map $\delta: \Sigma \to [0, \varepsilon)$, with $X = \delta^{-1}(0)$, which satisfies the conclusions of Lemma 4.2.

Consider then an embedding e of Σ in S^n such that $\rho(z, e(z)) \leq \delta(z)$ for each z in Σ and $e(\Sigma)$ is locally flat modulo $e(X) = X$. The restrictions of Lemma 4.2 insure that each component U^* of $S^n - e(\Sigma)$ is 1-ULC in $U^* \cup (F \cap X)$, where $F \cap X$ is a σ-compactum contained in $F - \bigcup C_i$, yielding $\text{dem}_{e(\Sigma)} F \cap X = \text{dem}_\Sigma F \cap X \leq 1$. According to Corollary 3.4, $e(\Sigma)$ bounds two crumpled cubes of Type 1.

5. A SHRINKING THAT RESPECTS NULL SEQUENCES

The first result in this section discloses a technical control useful for regulating the squeezing of some arc fibers in n-dimensional annuli. The proof, which will not be given here, retraces the rather lengthy argument for Theorem 5.1 of [14]; the reader who understands that argument should be able to fill in the supplementary details required.

Before stating the Shrinking Lemma we should explain the terminology that arises; a sewing h of crumpled n-cubes C_0 and C_1 has the *Mismatch Property* if and only if there exist sets F_j in Bd C_j such that $F_j \cup \text{Int}\, C_j$ is 1-ULC, $j = 0, 1$, and $h(F_0) \cap F_1 = \emptyset$. Pertinent information is given by Theorem 9.1 of [14], asserting that, in case C_0 and C_1 are of Type 1 and $n \geq 5$, $C_0 \cup_h C_1 = S^n$ if and only if h has the Mismatch Property.

LEMMA 5.1 (Controlled Shrinking Lemma). *Suppose* $\omega: S^{n-1} \times I \to S^n$ $(n \geq 5)$ *is an embedding;* X *is a compact subset of* S^{n-1} (*without loss of generality,* $\omega(S^{n-1} \times \{0, 1\})$ *is locally flat modulo* $\omega(X \times \{0, 1\})$); C_j *is the closed* n*-cell-complement in* $S^n - \omega(S^{n-1} \times 1/2)$ *bounded by* $\omega(S^{n-1} \times j)$, $j = 0, 1$; *the natural homeomorphism* $\omega(\langle s, 0\rangle) \to \omega(\langle s, 1\rangle)$ *from* Bd C_0 *to* Bd C_1 *has the Mismatch Property; and* $\{Z_i\}_{i=1}^{\infty}$ *is a null sequence of compact subsets of* $S^n - \omega(X \times I)$.

Then for each $\varepsilon > 0$ *and each open set* U *containing* $\omega(X \times I)$ *there exists a homeomorphism* Θ *of* S^n *to itself such that*

(i) $\Theta | S^n - U = 1 | S^n - U$,

(ii) $\operatorname{diam} \Theta\omega(x \times I) < \varepsilon$ *for each* $x \in X$,

(iii) *for each point* $p \in S^n$ *such that* $\Theta(p) \neq p$, *there exists* $x_p \in X$ *such that* $\{\Theta(p), p\} \subset N_\varepsilon(\omega(x_p \times 1))$, and

(iv) *either* $\Theta | Z_i = 1 | Z_i$ or $\operatorname{diam} \Theta(Z_i) < \varepsilon$ *for* $i = 1, 2, \cdots$.

The Controlled Shrinking Lemma has direct application to a decomposition theorem.

THEOREM 5.2. *Suppose that for each* $i = 1, 2, \cdots$ *there exist an embedding* $\omega_i : S^{n-1} \times I \to S^n$ $(n \geq 5)$ *and a compact set* X_i *in* S^{n-1} *satisfying the hypotheses of Lemma* 5.1 *and such that* $\{\omega_i(X_i \times I)\}_{i=1}^{\infty}$ *forms a null sequence of pairwise disjoint sets. Let* G *denote the upper semicontinuous decomposition of* S^n *whose set of non-degenerate elements is* $\{\omega_i(x \times I) : x \in X_i, i = 1, 2, \cdots\}$. *Then the decomposition space* S^n/G *is homeomorphic to* S^n.

Proof. Fix a positive number ε. There exists a large positive integer K such that $k \geq K$ implies diameter $\omega_k(X_k \times I) < \varepsilon$. Choose pairwise disjoint open sets U_j containing $\omega_j(X_j \times I)$, $j = 1, \cdots, K$. Apply the Controlled Shrinking Lemma for $j = 1, \cdots, K$, with $\{Z_i = \omega_{K+i} \times I)\}_{i=1}^{\infty}$, to shrink the arcs of $\omega_j(X_j \times I)$ to size less than $\varepsilon/2$ via a homeomorphism Θ that is the identity outside $\bigcup U_j$. Then for each $g \in G$, $\operatorname{diam} \Theta(g) < \varepsilon$, and, in addition, it is easily verified that there exists g' (depending on g) such that $g \cup \Theta(g) \subset N_\varepsilon(g')$. This establishes a variation to what is often called Bing's Shrinking Criterion, introduced by R. H. Bing and studied in related form by many others, and implies that S^n/G is homeomorphic to S^n (*cf.* the discussions of [19, pp. 287-288] and [8, p. 92]).

6. REEMBEDDINGS OF CRUMPLED n-CUBES

THEOREM 6.1 (Reembedding Theorem). *Let* C *denote a crumpled* n-*cube in* S^n $(n \geq 5)$. *For each* $\varepsilon > 0$ *there exists an embedding* h *of* C *in* S^n *such that* $\rho(h, 1|C) < \varepsilon$ *and* $S^n - h(\operatorname{Int} C)$ *is an* n-*cell.*

Imitating the proof for the 3-dimensional case, we derive this result through the Main Lemma, stated below.

6.2. MAIN LEMMA. *Let* C *denote a crumpled* n-*cube in* S^n $(n \geq 5)$ *with* $\Sigma = \operatorname{Bd} C$, *and let* ε *denote a positive number. Then there exists an embedding* f *of* Σ *in* S^n *such that* $\rho(f, 1|\Sigma) < \varepsilon$ *and there exists an embedding* h *of* C *in* S^n *such that* $\rho(h, 1|C) < \varepsilon$ *and* $h(C) \cap f(\Sigma) = \emptyset$.

For a detailed argument establishing Theorem 6.1 as a consequence of the Main Lemma, see Theorem 2 of [17]. The idea can be simply expressed: embed C as the limit h of a sequence $\{h_i\}$ of embeddings provided by the lemma; simultaneously obtain a sequence $\{f_i\}$ of embedings of $\Sigma \doteq \operatorname{Bd} C$ homeomorphically within 1/i of $h_{i-1}(\operatorname{Bd} C)$; design epsilonic controls to yield not only that $h(C) \cap f_i(\Sigma) = \emptyset$ for all i but also that $\rho(f_i, h|\operatorname{Bd} C) < 2/i$. It will then follow that $S^n - h(C)$ is 1-ULC, implying that the closure of $S^n - h(C)$ is an n-cell ([1] or [18, Theorem 5]).

A fundamental step in the proof of the Main Lemma involves an application of the following weak form of the Main Theorem:

6.3. WEAK FORM OF REEMBEDDING THEOREM 6.1. *Suppose the* (n - 1)-*sphere* Σ *in* S^n $(n \geq 5)$ *bounds two crumpled* n-*cubes* K_0 *and* K_1 *of Type* 1. *Then for each* $\varepsilon > 0$ *there exists an embedding* h *of* K_0 *in* S^n *such that* $\rho(h, 1|K_0) < \varepsilon$ *and* $S^n - h(\operatorname{Int} K_0)$ *is an* n-*cell.*

Bryant, Edwards, and Seebeck [5] and Štan'ko [24] essentially have proved Theorem 6.3, with arguments that would establish much more general versions than the weak form stated here. For completeness we provide, in an appendix, still another proof of 6.3, closely aligned in spirit with the proof given in this section, but avoiding most of its complexities.

Proof of the Main Lemma. The crux of the argument depends upon a somewhat elaborate construction of a very special cellular decomposition G of S^n. It is built up in two stages, the first based upon Ancel and Cannon's locally flat approximation theorem and upon Price and Seebeck's result that sufficiently close locally flat approximations are ambient isotropic, and the second based upon Theorem 6.3.

A preliminary decomposition. Set $\varepsilon' = \varepsilon/12$. Then there exists a Sierpiński curve X in Σ such that X is standardly embedded in Σ and the components of $\Sigma - X$ have diameter less than ε', and there exists a continuous function $\delta: \Sigma \to [0, \varepsilon')$ with the property stated in Lemma 4.3. Let V denote an open subset of S^n such that $V \cap \Sigma = \Sigma - X$. By the locally flat approximation theorem of [1], there exists an embedding e of Σ in $\Sigma \cup V$ such that $e|X = 1$, $\rho(z, e(z)) \leq \delta(z)$ for each point z of Σ, and $e(\Sigma)$ is locally flat modulo $e(X) = X$. In addition, it follows from Theorem 3 of [18] that e can be chosen so close to the inclusion that there exists a pseudoisotopy Θ_t of S^n onto itself such that Θ_0 = identity,

$$\Theta_t|S^n - V = 1|S^n - V,$$

$\rho(\Theta_0, \Theta_t) < \varepsilon'$ and $\Theta_1 e = 1|\Sigma$; furthermore, by brute force or by [19, Theorem A] one can require that every nondegenerate inverse set $\Theta_1^{-1}(p)$ intersects $e(\Sigma - X)$. Define a decomposition G' of S^n as $G' = \{\Theta_1^{-1}(p): p \in S^n\}$, and note that $\operatorname{diam} g' < 2\varepsilon'$ for each $g' \in G'$.

The decomposition G. According to Lemma 4.3, $e(\Sigma)$ bounds crumpled n-cubes C_0^* and C_1^* of Type 1, with the notation arranged so that $\Theta_1(C_0^*) = C$. By Theorem 6.3, there exists an ε'-homeomorphism v_0 of C_0^* in S^n such that $S^n - v_0(\operatorname{Int} C_0^*)$ is an n-cell. Similarly, there exists an embedding v_1 of C_1^* in $S^n - v_0(C_0^*)$ such that $S^n - (v_0(\operatorname{Int} C_0^*) \cup v_1(\operatorname{Int} C_1^*))$ is homeomorphic to $\Sigma \times I$; in particular, one can describe a homeomorphism $\omega: \Sigma \times I \xrightarrow{\text{onto}} S^n - (v_0(\operatorname{Int} C_0^*) \cup v_1(\operatorname{Int} C_1^*)$ such that the composition

$$\Sigma \longrightarrow \Sigma \times j \xrightarrow{\ \omega\ } \omega(\Sigma \times j) = v_j(\operatorname{Bd} C_j^*) \xrightarrow{(v_j|e(\Sigma))^{-1}} e(\Sigma)$$

coincides with e' for $j = 0, 1$. A significant feature here is that, although v_0 can be obtained close to the inclusion, one cannot expect a priori to obtain simultaneously v_1 close enough to the inclusion so as to have the fibers of ω be very small.

Define $C_j = v_j(C_j^*)$ for $j = 0, 1$. Define a map m of S^n onto S^n in the obvious manner to squeeze out the arc fibers of the annulus $\omega(\Sigma \times I)$; that is, $m|C_j = v_j^{-1}$ for $j = 0, 1$ and $m\omega(z \times I) = e(z)$ for $z \in \Sigma$. Now define the decomposition G of S^n as $G = \{(\Theta_1 m)^{-1}(p): p \in S^n\}$. The nondegenerate elements of G consist of arcs $\omega(x \times I)$ where $x \in X$ and other sets of the form

$$\omega(z \times I) \cup v_0(g' \cap C_0^*) \cup v_1(g' \cap C_1^*),$$

where $g' \cap e(\Sigma) = e(z)$ for some $z \in \Sigma - X$ and $g' \in G'$. Recalling that v_0 moves points of C_0^* less than ε', one sees that $\operatorname{diam} v_0(g' \cap C_0^*) < 4\varepsilon'$ for each $g' \in G$ and that, as a result, $\operatorname{diam} (g \cap C_0) < 4\varepsilon'$ for each $g \in G$.

The embedding f. Since $m|C_0 = v_0^{-1}|v_0(C_0^*)$ is an ε'-embedding, there exists $r \in (0, 1)$ such that the composition

$$\omega(\Sigma \times r) \longrightarrow \omega(\Sigma \times 0) \xrightarrow{m|\mathrm{Bd}\, C_0} e(\Sigma) \xrightarrow{e^{-1}} \Sigma$$

is a $3\varepsilon'$-homeomorphism. The required embedding f of Σ in S^n is simply the inverse.

The modified decomposition G^*. Enumerate the closures of the components of $\Sigma - X$ as $D_1, D_2, \cdots$. In the n-cell $B = S^n - \mathrm{Int}\, C_0$ thicken the cells $\omega(D_i \times 0)$ slightly to produce a null sequence of $\{B_i\}$ of n-cells satisfying

$$B_i \subset \omega(\Sigma \times [0, r)) \subset S^n - f(\Sigma),$$

$$B_i \cap \mathrm{Bd}\, C_0 = B_i \cap \omega(\Sigma \times 0) = \omega(D_i \times 0),$$

$$B_i \cap B_j = \emptyset, \quad i \neq j,$$

$$\mathrm{diam}\, B_i < 5\varepsilon',$$

$$(B, B_i) \text{ is homeomorphic to } (B^n, B^n_+).$$

This final condition, which carries the usual connotation that some homeomorphism of B to B^n takes B_i onto B^n_+, resolves the insistence woven into Section 4 that X be standardly embedded in Σ; by way of application, one can construct manually homeomorphisms ϕ_i of S^n onto itself such that for $i = 1, 2, \cdots, \phi_i|C_0 = 1|C_0$ and $\phi_i(g \cap \mathrm{Int}\, B) \subset \mathrm{Int}\, B_i$ for each $g \in G$ such that $g \cap \mathrm{Bd}\, C_0 \subset \omega(D_i \times 0)$.

Let G^* denote the upper semicontinuous decomposition of S^n having for its nondegenerate elements the set $\bigcup_i \{\phi_i(g)$: g is a nondegenerate element of G and $g \cap \mathrm{Bd}\, C_0 \subset \omega(D_i \times 0)\}$.

The equivalence of S^n *and* S^n/G^*. Consider the decomposition G^{**} of S^n whose set of nondegenerate elements if $\bigcup_i \{\phi_i\, \omega(z \times I): z \in D_i\}$. According to the application (Theorem 5.2) of the Controlled Shrinking Lemma, the decomposition space S^n/G^{**} is homeomorphic to S^n. Let $F: S^n \to S^n$ denote a map realizing this equivalence and let $F(G^*)$ denote the decomposition $F(G^*) = \{F(g^*): g^* \in G^*\}$. Here it is important to recognize that each nondegenerate element of G^{**} is contained in an element of G^*. Clearly then $S^n/F(G^*)$ and S^n/G^* are equivalent. Furthermore, $S^n/F(G^*)$ is related to our preliminary decomposition space S^n/G', as follows: for $i = 1, 2, \cdots$ there exists an open set V_i' in V, containing all of the nondegenerate elements of G' that meet $e(D_i)$, such that V_i' and V_j' are disjoint, $i \neq j$, and $\phi_i m^{-1}(V_i') \subset C_0 \cup \mathrm{Int}\, B_i$. Each $F\phi_i m^{-1}|V_i'$ is an embedding of V_i' in S^n, all of which generate a composite embedding $F\phi m^{-1}|V'$ of $V' = \bigcup V_i'$ in S^n that induces a bijective correspondence between the nondegenerate elements of $F(G^*)$ and that extends to an embedding of $e(\Sigma) \cup V'$ into S^n. Thus, $F(G^*)$ restricts to a cellular decomposition of $F\phi m^{-1}(V')$ that yields a manifold, and, because the nondegenerate elements get small near the frontier of $F\phi m^{-1}(V')$, Theorem A of [19] (see also [8, Theorem 62]) implies that $S^n/F(G^*)$ is homeomorphic to S^n. Therefore, S^n/G^* and S^n are equivalent.

The embedding h. Observe that for each nondegenerate element g^* of G^* there exists an integer i such that $g^* \subset (g^* \cap C_0) \cup B_i \subset N_{5\varepsilon'}(g^* \cap \mathrm{Bd}\, C_0)$, implying that diam $g^* < 10\varepsilon'$, and recall that restrictions on B_i imply $g^* \cap f(\Sigma) = \emptyset$. By Statement 1 of [19, p. 287] or, alternatively, Theorem 62 of [8], there exists a $10\varepsilon'$-map π of S^n onto itself such that $\pi | f(\Sigma)$ = identity and $G^* = \{\pi^{-1}(p): p \in S^n\}$. The required embedding h of C in S^n is defined as $h = \pi(\theta_1 m | C_0)^{-1}$ Then

$$\rho(h, 1_C) < 12\varepsilon' = \varepsilon,$$

completing the proof.

COROLLARY 6.4. *For* $n \geq 5$ *each crumpled* n-*cube is a closed* n-*cell-complement.*

COROLLARY 6.5. *For* $n \geq 5$ *the* n-*cell* B^n *is a universal crumpled* n-*cube; that is, for any crumpled* n-*cube* C *and any sewing* s *of* C *and* B^n, $C \cup_s B^n$ *is homeomorphic to* S^n.

COROLLARY 6.6. *Suppose that* B *is an* n-*cell in* S^n $(n \geq 5)$ *and* G *is an upper semicontinuous decomposition of* S^n *such that* S^n/G *is homeomorphic to* S^n *and each* $g \in G$ *intersects* Bd B *in at most one point. Then for the decomposition* $\tilde{G}$ *of* S^n *having as its nondegenerate elements the set*

$$\{g \cap (S^n - \text{Int } B): g \text{ is a nondegenerate element of } G\},$$

the decomposition space $S^n/\tilde{G}$ *is homeomorphic to* S^n.

At present, whether Corollary 6.6 holds when B is a crumpled n-cube rather than an n-cell is an unresolved problem.

COROLLARY 6.7. *Suppose that* C *is a crumpled* n-*cube and* $n \geq 5$. *Then there exists an upper semicontinuous decomposition* G *of the* n-*cell* B^n *such that each nondegenerate element* g *of* G *is a tame arc intersecting* S^{n-1} = Bd B^n *in an endpoint of* g *and such that* B^n/G *is homeomorphic to* C.

Proof. Assume that C is embedded in S^n so that S^n - Int C is an n-cell. Then there exists an embedding ω of Bd $C \times [0, 1]$ in S^n - Int C such that ω(Bd $C \times 0$) is Bd C. Thus, $C \cup \omega(\text{Bd } C \times [0, 1/2])$ is homeomorphic to B^n and the nondegenerate elements of G correspond to the arcs $\omega(z \times [0, 1/2])$ for $z \in$ Bd C.

COROLLARY 6.8. *Suppose* C *is a crumpled* n-*cube in* E^n $(n \geq 5)$. *Then* $E^n/C \times E^1$ *is homeomorphic to* E^{n+1}.

Proof. By Theorem 6.1 there exists an n-cell B in E^n such that E^n/B is equivalent to E^n/C. The corollary then follows from work of Bryant [4].

APPENDIX

The weak form of the Reembedding Theorem (6.3) follows immediately, as before, from a weak version of the Main Lemma.

LEMMA. *Let* Σ *denote an* (n - 1)-*sphere in* S^n $(n \geq 5)$ *that bounds two crumpled* n-*cubes* K_0 *and* K_1 *of Type* 1, *and let* ε *denote a positive number. Then there exists an embedding* f *of* Σ *in* S^n *such that* $\rho(f, 1|\Sigma) < \varepsilon$ *and there exists an embedding* h *of* K_0 *in* S^n *such that* $\rho(h, 1|K_0) < \varepsilon$, $h(K_0) \cap f(\Sigma) = \emptyset$, *and* $S^n - h(\text{Int } K_0)$ *is of Type* 1.

Proof. Choose a triangulation T of Σ of mesh less than $\varepsilon' = \varepsilon/6$. By hypothesis Σ contains an F_σ set F such that $\text{dem}_\Sigma F = 0$ and $F \cup \text{Int } K_j$ is 1-ULC, j = 0, 1. Then after a small adjustment of T in Σ one can assume that

$$T^{(n-2)} \cap F = \emptyset.$$

Let $\delta\colon \Sigma \to [0, \varepsilon')$ be a continuous function for which $T^{(n-2)} = \delta^{-1}(0)$, satisfying the conclusions of Lemma 4.2, and let V denote an open subset of S^n for which $V \cap \Sigma = \Sigma - T^{(n-2)}$. There exists an embedding e of Σ in S^n such that

$$\rho(z, e(z)) \leq \delta(z) \quad \text{for each } z \text{ in } \Sigma$$

and $e(\Sigma)$ is locally flat modulo $e(T^{(n-2)}) = T^{(n-2)}$ [1], and by [18, Theorem 3] e can be chosen sufficiently close to the inclusion that there exists a pseudoisotopy Θ_t of S^n onto itself such that $\Theta_0 = 1$, $\Theta_t | S^n - W = 1 | S^n - W$, $\rho(\Theta_0, \Theta_t) < \varepsilon'$, $\Theta_1 e = 1 | \Sigma$ and $\Theta_1^{-1}(p)$ is nondegenerate only if $\Theta_1^{-1}(p)$ intersects $e(\Sigma - T^{(n-2)})$. Define a decomposition G of S^n as $\{\Theta_1^{-1}(p): p \in S^n\}$. Note that diam $g < 2\varepsilon'$ for each $g \in G$.

The sphere $e(\Sigma)$ is embedded such that $S^n - e(\Sigma)$ is 1-ULC (Lemma 4.2), and thus $e(\Sigma)$ bounds n-cells B' and B [9, 11, 18], where $\Theta_1(B') = K_0$ and $\Theta_1(b) = K_1$. The required locally flat embedding f of Σ in S^n is just e followed by an ε'-homeomorphism of $e(\Sigma) = \text{Bd } B$ onto a flat sphere in Int B.

Enumerate the closures of the components of $\Sigma - T^{(n-2)}$ as $D_1, \cdots, D_k$, and note that diam $e(D_i) < 3\varepsilon'$. Determine n-cells $B_1, \cdots, B_k$ in B such that

$$B_i \subset B - f(\Sigma),$$

$$B_i \cap \text{Bd } B = e(D_i),$$

$$B_i \cap B_j = e(D_i \cap D_j), \qquad i \neq j,$$

$$\text{diam } B_i < 3\varepsilon',$$

$$(B, B_i) \text{ is homeomorphic to } (B^n, B^n_+).$$

As before, for $i = 1, \cdots, k$ there exists a homeomorphism ϕ_i of S^n to itself such that $\phi_i | B' = 1 | B'$ and

$$\phi_i(g) \cap \text{Int } B \subset \text{Int } B_i \quad \text{for each } g \in G \text{ such that } g \cap e(\Sigma) \subset e(D_i).$$

Let G^* denote the decomposition of S^n having as its nondegenerate elements the set $\bigcup_{i=1}^{k} \{\phi_i(g)\colon g$ is a nondegenerate element of G and $g \cap e(\Sigma) \subset e(D_i)\}$. Note that for each nondegenerate $g^* \in G^*$, diam $g^* < 5\varepsilon'$ and $g^* \subset S^n - f(\Sigma)$. Again S^n/G^* is homeomorphic to S^n ([19, Theorem A] or [8, Theorem 62]); moreover, there exists a $5\varepsilon'$-map π of S^n to itself such that $\pi | f(\Sigma) = 1 | f(\Sigma)$ and

$$G^* = \{\pi^{-1}(p): p \in S^n\}.$$

The required embedding of K_0 in S^n is defined as $h = \pi(\Theta_1 | B')^{-1}$.

The proof that the crumpled n-cube $S^n - h(\text{Int } K_0)$ is of Type 1 is left to the reader; the key to showing this is that h as defined extends over a neighborhood of $K_0 - T^{(n-2)}$ in S^n.

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