ON VANISHING EICHLER PERIODS AND CARLESON SETS

Thomas A. Metzger

1. INTRODUCTION

Let Γ be a Fuchsian group acting on the unit disk D in the complex plane, and let q be an integer, $q \geq 2$. An analytic function f defined on D is said to be an *automorphic form* of weight q with respect to Γ if $(f \circ \gamma)\gamma'^q = f$ for all γ in Γ.

The *Bers spaces* $A_q^p(\Gamma)$, $1 \leq p \leq \infty$, are defined as those Banach spaces of analytic automorphic forms of weight q such that

$$\|f\|_q^p = \iint_{D/\Gamma} |f(z)|^p (1 - |z|^2)^{pq-2}\, dx\, dy < \infty, \qquad 1 \leq p < \infty;$$

$$\|f\|_\infty = \sup_D |f(z)|(1 - |z|^2)^q < \infty, \qquad p = \infty.$$

Any analytic automorphic form f of weight q can be integrated $(2q - 1)$ times to get an analytic function $h \equiv I^{2q-1} f$ which satisfies

$$(h \circ \gamma)\gamma'^{1-q} = h + c(\gamma, f) \qquad \text{for all } \gamma \text{ in } \Gamma.$$

This $c(\gamma, f)$ is a polynomial of degree $s \leq 2q - 2$ and is called the *Eichler period* of f along γ. Bers [2] proved

THEOREM A. *If* Γ *is a group of the first kind, and the Eichler period of* ϕ *in* $A_q^\infty(\Gamma)$ *vanishes for all* γ *in* Γ, *then* $\phi \equiv 0$.

We shall extend this to say that if there exists a ϕ in $A_q^p(\Gamma)$ with vanishing Eichler period for all γ in Γ, then either $\phi \equiv 0$ or the limit set L is sparse in a special sense; *i.e.*, L is a Carleson set.

Conversely, Pommerenke [10] has recently shown that if L is a Carleson set, then there exists an f_0 in $A_2^\infty(\Gamma)$ such that $c(\gamma, f_0) = 0$ for all γ in Γ. I wish to thank Professor Pommerenke for our discussions on this topic. Also, I wish to thank the referee for pointing out a gap in the original proof of Theorem 1.

2. PRELIMINARIES

A closed set E of Lebesgue measure zero contained in ∂D is said to be a *Carleson set* if in the canonical representation of its complement $\partial D \setminus E$ as a countable union of disjoint open intervals I_n, the lengths $\ell(I_n)$ satisfy

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$$\sum_{n=1}^{\infty} \ell(I_n) \log \ell(I_n) > -\infty .$$

As is well known, these sets are the zero-sets of Lip α functions (see [2]) where $g \in$ Lip α if and only if $|g'(z)| = O((1 - |z|^2)^{\alpha-1})$, $0 < \alpha \leq 1$. We shall also consider the spaces of analytic functions Lip(α, p) consisting of those g such that

$$\left\{ \int_0^{2\pi} |g'(re^{i\theta})|^p d\theta \right\}^{1/p} = O((1 - r)^{\alpha-1}) .$$

We shall need the result of Caveny and Novinger [4] that $f \in$ Lip$(1, p)$, $1 \leq p \leq \infty$, implies $Z(f) = \{\zeta \in \partial D: f(\zeta) = 0\}$ is a Carleson set.

We conclude by noting:

LEMMA 1. *Let* k *be a continuous function on* $\overline{D}$ *and suppose* $(k \circ \gamma)\gamma'^{1-q} = k$ *for all* γ *in* Γ. *Then* $k(\zeta) \equiv 0$ *for all* ζ *in* L.

The proof follows upon noting that $\gamma(0)$ clusters at ζ and $\gamma'^{q-1}(0) \to 0$, and so the continuity of k yields the result.

The following estimates on the mean growth and Taylor coefficients of f in $A_q^p(\Gamma)$ will be necessary in our proof of Theorem 1.

LEMMA 2. *Let* Γ *be any Fuchsian group and suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is in* $A_q^p(\Gamma)$. *Then*

(i) $A_k = O(k^q)$;

(ii) $M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} = O((1 - r)^{-q})$;

(iii) $|f(z)| = O((1 - |z|)^{-q-1/p}$.

Proof. (i) will appear in Lehner [8], and (iii) follows immediately from (ii). Hence it suffices to prove (ii). Let $n(r, z)$ be the number of images of z under Γ which lie in the set $D_r = \{z: |z| < r\}$. It is well known that $n(r, z) \leq C(1 - r)^{-1}$ for all z in D. Let Ω be a fundamental region for Γ, and define $\Omega_r = D_r \cap \Omega$. Then

$$(1 - r)^{pq-1} M_p^p(r^2, f) \leq C_1 \int_{r^2}^{r} (1 - t^2)^{pq-2} M_p^p(t, f)\, t\, dt$$

$$\leq C_2 \iint_{D_r} (1 - |z|^2)^{pq-2} |f(z)|^p \,dx\,dy .$$

Since $D_r \subset \bigcup_{\gamma \in \Gamma} \gamma\Omega_r$ and $|f(z)|^p(1 - |z|^2)^{pq}$ is Γ-invariant, it follows that

$$(1 - r)^{pq-1} M_p^p(r^2, f) \leq C \iint_{\Omega_r} n(r, z)(1 - |z|^2)^{pq-2} |f(z)| \, dx\, dy$$

$$\leq C_2 C(1 - r)^{-1} \iint_{\Omega} |f(z)|(1 - |z|^2)^{pq-2} \, dx\, dy$$

$$= C_3(1 - r)^{-1} \|f\|_p^p,$$

and the proof of (ii) is complete.

If, moreover, Γ is of convergence type, then it follows that

$$\sum_{\gamma \in \Gamma} (1 - |\gamma z|^2) \leq M,$$

so that

$$\iint_D |f(z)|^p (1 - |z|^2)^{pq-1} \, dx\, dy \leq M \|f\|_p^p \tag{1}$$

Inequality (1) follows from the fact that $D = \bigcup_{\gamma \in \Gamma} \gamma\Omega$, where the 2-dimensional measure of $\partial\Omega$ is zero, and from the fact that $|f(z)|^p (1 - |z|^2)^{pq}$ is Γ-invariant (see [9] for complete details). It is (1) which will enable us to conclude that $I^3 f$ is in Lip(1, 1) (*i.e.*, when $f \in A_2^1(\Gamma)$), which seems to be the major difficulty in the proof.

3. THE MAIN RESULT

We now assert:

THEOREM 1. *Let* $f \in A_q^p(\Gamma)$, $1 \leq p \leq \infty$, $q \geq 2$, *and assume that the Eichler period of* f *vanishes for each* γ *in* Γ. *Then either* $f \equiv 0$ *or* L *is a Carleson set.*

Proof. We first show that if Γ is of divergence type and if F in $A_q^p(\Gamma)$ has vanishing Eichler periods, then $h \equiv I^{2q-1} f$ is identically zero. If

$$f(z) = \sum_{k=0}^{\infty} A_k z^k$$

then $A_k = O(k^q)$ and $h \in H^2(D)$, the Hardy class. Let $h^*(\zeta) = \lim_{r \to 1} h(r\zeta)$ for each $\zeta \in \partial D$. Since Γ is of divergence type, it follows that Γ is of the first kind; *i.e.*, every point of ∂D is in the limit set. Moreover (see [5]), the set of transitive points has measure 2π. For each transitive point ζ, there is a sequence of $\gamma_n \in \Gamma$ such that $\gamma_n(0) \to \zeta$ inside any Stolz angle. Since $h(\gamma_n(0)) = h(0)\gamma_n'^{q-1}(0)$, $\gamma_n'^{q-1}(0) \to 0$, and $h(\gamma_n(0)) \to h^*(\zeta)$ for almost every transitive point ζ, it follows that $h^* \equiv 0$. Thus h and, of course, f must vanish identically.

We now turn to the case where Γ is of convergence type. If $f \not\equiv 0$, then it suffices to show that $h \equiv I^{2q-1} f$ belongs to Lip α for some $\alpha > 0$, or to Lip(1, p). This is sufficient because analytic functions in Lip α or Lip(1, p) are continuous on

$\overline{D}$ and every closed subset of a Carleson set is again a Carleson set ($x \log(1/x)$ is a decreasing function for $x < 1/e$). Thus, if $h \equiv I^{2q-1} f$ is in Lip α or Lip (1, p) and has vanishing Eichler periods, Lemma 1 implies that $L \subset Z(h)$ (the zero-set of h) and thus L is a Carleson set.

Since $f \in A_q^p(\Gamma)$ implies that $|f(z)| = O((1 - |z|^2)^{-q-1/p})$, it follows that h belongs to Lip 1 if $q > 2$. If $q = 2$ and $p > 1$, then Lemma 2(ii) implies that h belongs to Lip(α, p) for all $\alpha < 1$ and thus h belongs to Lip$(\alpha - 1/p)$.

Hence the theorem is proved except in the case $p = 1$, $q = 2$. To handle this case, we shall show substantially more about h in certain cases. In particular, we shall prove that h belongs to Lip (1, 1) if $f \in A_2^1(\Gamma)$, and thus the result of [4] cited above will complete the proof of the theorem.

LEMMA 3. *Let* Γ *be a group of convergence type and* $f \in A_q^p(\Gamma)$, $1 \leq q < \infty$, $1 \leq p \leq 2$. *Then* $I^{q+1} f \in$ Lip (1, p).

Proof. It suffices to show $I^q f \in H^p(D)$, the Hardy class. In order to do this, we define the *Bessel potential operator* J^t (see [6]) by

$$J^t\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} (n+1)^{-t} a_n z^n .$$

Since $J^q f$ is in $H^p(D)$ if and only if $I^q f$ is in $H^p(D)$, we need only apply Theorem 5(iii) of [6] to $J^q f$. This asserts that $J^q f$ is in $H^p(D)$ if

$$\iint_D |f(z)|^p (1 - |z|^2)^{pq-1}\, dx\, dy < \infty .$$

But this is precisely (1), so by the remarks at the end of Section 2, the proof of Lemma 3 is complete.

Remarks. (i) Lemma 3 allows us to improve some of the results on the Taylor coefficients of $f \in A_q^p(\Gamma)$ presented in [9].

(ii) A similar proof using [7] enables one to show that $f \in A_q^p(\Gamma)$, $1 \leq q < \infty$, $2 \leq p < \infty$, implies $I^{q+1} f \in$ Lip(α, p), $0 < \alpha < 1$. However, the case $\alpha = 1$ is still open.

(iii) If one assumes the existence of factors of automorphy, then one can allow q to be arbitrary ($q \geq 1$) and Lemma 3 is still valid.

It should also be noted that Theorem 1 fails completely if $q = 1$, for if $f \in A_1^2(\Gamma)$ and the Eichler period of f along each γ in Γ vanishes, then $h = I^1 f$ in an automorphic function on the associated Riemann surface $W = D/\Gamma$. In [11], Pommerenke has constructed a group Γ of first kind with a nonconstant g in AD(W), so that $g' \in A_1^2(\Gamma)$ and g' has vanishing Eichler period along each γ in Γ.

4. FUCHSIAN GROUPS WHOSE LIMIT SET IS A CARLESON SET

In response to a question by C. J. Earle, we give here a sufficient condition on Γ for L to be a Carleson set, and then verify that all finitely generated groups of the second kind satisfy the condition. It should be noted that Ch. Pommerenke has proved the corollary below by a different method. However, his proof does not seem to be applicable to the infinitely generated case, whereas ours, presumably, does apply in this case. If Ω is the Ford fundamental region for Γ, define $\overline{E} = \partial\Omega \cap \partial D$ and denote the Lebesgue measure of a set F in ∂D by $m(F)$.

THEOREM 2. *Let* Γ *be a Fuchsian group with* $m(L) = 0$. *Suppose further that* $\sum_{\gamma\in\Gamma} \ell(\gamma(\overline{E})) \log[2\pi/\ell(\gamma(\overline{E}))]$ *converges, where* $\ell(\gamma\overline{E})$ *is the linear measure of* $\gamma(\overline{E})$. *Then* L *is a Carleson set.*

Proof. Since $m(L) = 0$, one can replace $\overline{E}$ by $E = \overline{E}\backslash L$ and the series $\sum_{\gamma\in\Gamma} \ell(\gamma(E)) \log[2\pi/\ell(\gamma(E))]$ will again converge. Represent the open set $\mathscr{O} = \partial D\backslash L$ as the union of disjoint open intervals I_n. Breaking E into its components E_j, we see that for each i there exists a set Γ_i such that

$$I_i = \bigcup_{\gamma\in\Gamma_i} \gamma(E_{j(i)}).$$

Now $\gamma(E_{j(i)}) \cap \gamma(E_{k(i)}) = \emptyset$ for $k \neq j$, since E is a fundamental set for $\mathscr{O}$. Thus, we have

$$S \equiv \sum_{i=1}^{\infty} \ell(I_i) \log \frac{2\pi}{\ell(I_i)} = \sum_{i=1}^{\infty} \sum_{\gamma\in\Gamma_i} \gamma(E_{j(i)}) \log \frac{2\pi}{\ell(I_i)}.$$

But $\ell(I_i) \geq \ell(\gamma(E_{j(i)})$ for each i and j, so that $\log \frac{2\pi}{\ell(I_i)} \leq \log \frac{2\pi}{\ell(\gamma(E_{j(i)}))}$. Thus $S \leq \sum_{i=1}^{\infty} \sum_{\gamma\in\Gamma_i} \ell(\gamma(E_{j(i)})) \log \frac{2\pi}{\ell(\gamma(E_{j(i)}))}$. But E is the union of the $E_{j(i)}$, and thus we see that $S \leq \sum_{\gamma\in\Gamma} \ell(\gamma(E)) \log \frac{2\pi}{\ell(\gamma(E))}$. The proof is complete.

COROLLARY 1. *If* Γ *is a finitely generated group of the second kind, then* L *is a Carleson set.*

Proof. Since Γ is finitely generated and of the second kind, it follows that $m(L) = 0$. Moreover, there exists an $M = M(\Omega)$ such that the distance from ζ to L is larger than M for all ζ in $\overline{E} = \partial\Omega \cap \partial D$. Now, using the fact (see [1]) that there exists a $p = p(\Gamma) < 1$ such that $\sum_{\gamma\in\Gamma} (1 - |\gamma(0)|^2)^p < \infty$, we have for this p

$$\sum_{\gamma\in\Gamma} \ell(\gamma(\overline{E}))^p = \sum_{\gamma\in\Gamma} \left\{ \int_{\gamma(\overline{E})} |d\zeta| \right\}^p$$

$$= \sum_{\gamma\in\Gamma} \left\{ \int_{\overline{E}} |\gamma'(\zeta)|\, |d\zeta| \right\}^p \leq \sum_{\gamma\in\Gamma} M(1 - |\gamma(0)|^2)^p m(\overline{E}) < \infty .$$

Since $\sum_{\gamma\in\Gamma} \ell(\gamma(\overline{E})) \log[2\pi/(\gamma(\overline{E}))] \leq \sum_{\gamma\in\Gamma} (\ell(\gamma(\overline{E})))^p$, the proof is complete.

REFERENCES

1. A. F. Beardon, *Inequalities for certain Fuchsian groups*. Acta Math. 127 (1971), 221-258.

2. L. Bers, *Automorphic forms and Poincaré series for infinitely generated Fuchsian groups*. Amer. J. Math. 87 (1965), 196-214.

3. L. Carleson, *Sets of uniqueness for functions regular in the unit circle*. Acta Math. 87 (1952), 325-345.

4. D. J. Caveny and W. P. Novinger, *Boundary zeros of functions with derivative in* H^p. Proc. Amer. Math. Soc. 25 (1970), 776-780.

5. C. Constantinescu, *Über die Klassifikation der Riemannschen Flächen*. Acta Math. 102 (1959), 47-78.

6. T. M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*. J. Math. Anal. Appl. 38 (1972), 746-765.

7. ———, *Lipschitz spaces of functions on the circle and the disc*. J. Math. Anal. Appl. 39 (1972), 125-158.

8. J. Lehner, Proc. of London Math. Soc. Instructional Conference at Cambridge, England, (1974), to appear.

9. T. A. Metzger, *On the growth of the Taylor coefficients of automorphic forms*. Proc. Amer. Math. Soc. 39 (1973), 321-328.

10. Ch. Pommerenke, *On automorphic forms and Carleson sets*. Michigan Math. J. 23 (1976), 129-136.

11. ———, *On the Green's function of Fuchsian groups*. Ann. Acad. Sci. Fenn., to appear.

Department of Mathematics
University of Pittsburgh
Pittsburgh, Pennsylvania 15260