RATIONAL APPROXIMATION AND SWISS CHEESES

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1. INTRODUCTION

For the purpose of this note, an open set A of the complex plane will be called *regular* if its boundary ∂A is a finite union of piecewise C^2 simple closed curves. A compact set X of the plane will be called *areally disconnected* if for each $\alpha > 0$ there exist a finite number of pairwise disjoint regular open sets $A_1, \cdots, A_n$ (depending on α) for which $X \subset \bigcup_{k=1}^{n} A_k^-$ and, for $k = 1, \cdots, n$, $m_1(\partial A_k \cap X) = 0$ and $m_2(A_k \cap X) < \alpha$. (Here A_k^- denotes the closure of A_k, while m_1 and m_2 refer to Lebesgue arc length and planar measures.)

For any compact set X, let C(X) and R(X) denote, respectively, the algebra of continuous functions on X and the subalgebra of functions which are uniformly approximable on X by rational functions with poles off X. An obvious necessary condition in order that C(X) = R(X) is that the interior of X be empty. Various necessary and sufficient conditions for the validity of this equality are, in fact, known (*e.g.*, as consequences of Bishop's peak point criterion, Melnikov's peak point criterion, or Vitushkin's theorem; see Gamelin [4] or Zalcman [16]), but are not always easy to apply. A sufficient condition for C(X) = R(X) is contained in the following.

THEOREM 1. *If* X *is a compact set of the plane which is areally disconnected, then* C(X) = R(X).

COROLLARY 1 (Hartogs-Rosenthal [6]). *If* X *is a compact set of the plane and if* $m_2(X) = 0$, *then* C(X) = R(X).

The hypothesis $m_2(X) = 0$ implies that, for almost all real t, the line $\Re(z) = t$ intersects X in a set of zero linear measure. Hence, for each $\alpha > 0$, there exist a finite set of real numbers $t_0 < \cdots < t_n$ and a pair of real numbers $a < b$ satisfying $t_k - t_{k-1} < \alpha/(b-a)$ for $k = 1, \cdots, n$, with the properties that X is contained in the rectangle $(t_0, t_n) \times (a, b)$ and each segment $\{z: \Re(z) = t_k \text{ and } a \le \Im(z) \le b\}$, for $k = 0, \cdots, n$, intersects X in a set of zero linear measure. If the open rectangles $(t_{k-1}, t_k) \times (a, b)$, for $k = 1, \cdots, n$, are identified with the sets A_k considered at the beginning of this section, it is seen that X is areally disconnected.

It is clear from the above proof that Corollary 1 can be strengthened to the following.

COROLLARY 2. *If* X *is a compact set of the plane and if there exists a set of real numbers* $\{t\}$ *dense on the real line for which each of the vertical lines* $\Re(z) = t$ *intersects* X *in a set of zero linear measure, then* C(X) = R(X).

Obviously, the special role of the vertical direction in Corollary 2 is one of convenience, and any fixed direction would serve as well. It is noteworthy that, although the usual modern proofs of the Hartogs-Rosenthal theorem (see, *e.g.*, [4, p. 47] or [16, p. 110]), or even the original proof in [6], do not seem to yield Corollary

Received December 23, 1976.
This work was supported by a National Science Foundation research grant.

Michigan Math. J. 24 (1977).

2, nevertheless, a proof does occur, at least implicitly, in the paper of Alice Roth [11, pp. 98-100]. In fact, using an earlier lemma on p. 83, she shows that under the hypothesis of Corollary 2 above, the function $x = \Re(z)$, hence also $y = i(x - z)$, belongs to $R(X)$ and so, in view of the Weierstrass approximation theorem, $C(X) = R(X)$.

The proof of Theorem 1 will be given in Section 2 and will be based on some recent results on peak sets and subnormal operators. For later use, we recall here that a compact subset Q of a compact set X is a *peak set* of $R(X)$ if there exists a function f in $R(X)$ such that $f = 1$ on Q and $|f| < 1$ on $X - Q$. See, for instance, Gamelin [4]. An operator T on a Hilbert space $\mathfrak{H}$ is said to be *subnormal* if T has a normal extension N on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$. For some properties of such operators, see Halmos [5].

2. PROOF OF THEOREM 1

We shall need three lemmas.

LEMMA 1. *If* T *is a subnormal operator on a Hilbert space* $\mathfrak{H}$ *with the direct sum representation*

$$T = \sum_{k=1}^{m} \oplus T_k \quad \textit{on} \quad \mathfrak{H} = \sum_{k=1}^{m} \oplus \mathfrak{H}_k ,$$

then $\|T^*T - TT^*\| \leq \pi^{-1} \max_k \{m_2(\sigma(T_k))\}$, *where* $\sigma(T_k)$ *denotes the spectrum of* T_k *on* $\mathfrak{H}_k$.

Since a subnormal T is also hyponormal $(T^*T - TT^* \geq 0)$ and since $\|T^*T - TT^*\| = \max_k \{ \|T_k^*T_k - T_kT_k^*\| \}$, the result follows from Putnam [8].

LEMMA 2. *Let* X *be a compact set of the plane and let* C *be a simple closed curve which is piecewise* C^2 *and such that* $Q = (\text{ext } C)^- \cap X$ *is nonempty and* $C \cap X$ *has zero linear measure (the measure being that of arc length on* C*). Then* Q *is a peak set of* $R(X)$.

Lemma 2 is essentially contained in Lautzenheiser [7]. For completeness we give a demonstration below, which is a slightly modified version of his proof. If $Q = X$, Lemma 2 is trivial, so that it can be supposed that Q is a proper subset of X.

According to a result of Fatou, if Z is any set of zero arc length measure on the circle $|z| = 1$, then there exists a function f continuous on $|z| \leq 1$, analytic on $|z| < 1$, and such that $f = 0$ precisely on Z. Using this result, F. and M. Riesz [10, pp. 36-37], showed that there exists a function f continuous on $|z| \leq 1$, analytic on $|z| < 1$, and such that $f = 1$ on Z and $|f| < 1$ elsewhere. (In this connection, *cf.* Sz.-Nagy and Foiaş [13, p. 253].)

In view of the Riemann-Carathéodory mapping theorem, there exists a function $z = g(w)$ which maps $(\text{int } C)^-$ homeomorphically onto $|z| \leq 1$ and conformally on $|z| < 1$; *cf.* Rudin [12, p. 311]. Further, by a result of F. and M. Riesz [10], since C is rectifiable, sets of zero measure on C are mapped into sets of zero measure on $|z| = 1$ and conversely. In view of Mergelyan's theorem (see [4, p. 48]), $F = f(g(w))$ is the uniform limit on $(\text{int } C)^-$ of polynomials in w. (This last assertion is, in fact, a consequence of a much earlier theorem due to J. L. Walsh [14].) Consequently, if $Z = g(C \cap X)$ and if $Q' = (\text{int } C)^- \cap X$, then $F \mid Q' \in R(Q')$, $F = 1$

on $C \cap X$, and $|F| < 1$ on $Q' - C$. If F is defined by $F \equiv 1$ on Q, then clearly $F \mid Q \in R(Q)$ and $F \in C(X)$. According to a result of Vitushkin [15], since C is piecewise C^2, then C, hence also $C \cap X$, is analytically negligible (*cf.* Gamelin [4, p. 236]). It then follows from a result of Davie and Øksendal [3] that $F \in R(X)$, so that Q is a peak set of $R(X)$.

LEMMA 3. *Let* T *be subnormal on a Hilbert space and let* C *be a simple closed curve which is piecewise* C^2 *and such that both sets* $Q_1 = (\text{ext } C)^- \cap \sigma(T)$ *and* $Q_2 = (\text{int } C)^- \cap \sigma(T)$ *are nonempty, and such that* $C \cap \sigma(T)$ *has zero linear (arc length) measure. Then* T *is reducible and has the direct sum representation* $T = T_1 \oplus T_2$ *with* $\sigma(T_1) \subset Q_1$ *and* $\sigma(T_2) \subset Q_2$.

Lemma 3 was first proved by Lautzenheiser in [7] using Lemma 2 together with some results on subnormal operators which he obtained there. Another proof of Lemma 3, as a direct consequence of Lemma 2, follows from an application of Theorem 1 of Putnam [9].

The proof of Theorem 1 can now be completed as follows. If X is the spectrum of a subnormal operator T, then, in view of Lemma 3, the hypothesis of Theorem 1 readily implies that for every $\alpha > 0$ there exists a finite direct sum representation $T = \sum_{k=1}^{m} \oplus T_k$ on $\mathfrak{H} = \sum_{k=1}^{m} \oplus \mathfrak{H}_k$ with $m_2(\sigma(T_k)) < \alpha$. It follows from Lemma 1 that T must be normal, and this, in turn, implies that $C(X) = R(X)$. Indeed, if $C(X) \neq R(X)$, then by Bishop's peak point criterion (*cf.* [4, p. 54]), there would exist some x in X which is not a peak point of X. By an argument similar to that in Clancey and Putnam [1, p. 242], there would then exist a nonnormal subnormal operator S for which $x \in \sigma(S) \subset X$. If one chooses, say, any normal operator N with spectrum X, then $T = S \oplus N$ has spectrum X and is subnormal but not normal, in contradiction to the result proved above.

3. SWISS CHEESES

Let D denote the open unit disk $\{z: |z| < 1\}$ and let $\{D_k\}$, $k = 1, 2, \cdots$, be a sequence of open disks in D having pairwise disjoint closures and chosen so that $X = D^- - \bigcup D_k$ has an empty interior. Such a set X is called a Swiss cheese. If D_k has radius r_k and if $\sum r_k < \infty$, then it follows from Cauchy's integral theorem that $C(X) \neq R(X)$; see Gamelin [4, pp. 25-26]. As noted by Gamelin [4, p. 62], Swiss cheeses were first considered by Alice Roth [11]. It may also be observed that the preceding result, that $C(X) \neq R(X)$ whenever $\sum r_k < \infty$, was essentially proved by her; see [11, pp. 96-98].

Clearly, $\sum r_k^2 = 1$ is equivalent to the requirement that X have zero planar measure. As noted earlier, this is equivalent to the condition that almost all vertical lines intersect X in sets of zero linear measure, and the Hartogs-Rosenthal theorem implies that $C(X) = R(X)$. However, in view of Corollary 2, $C(X) = R(X)$ is assured even if only each member of a dense set of vertical lines intersects X in a set of zero linear measure. Further, it is easy to construct such Swiss cheeses X which have positive planar measure. To see this, choose a sequence of vertical lines $L_1, L_2, \cdots$, having intersections dense in $(-1, 1)$. Then remove the disks D_k centered on these lines in such a way that $\sum r_k^2 < 1$ and $X = D^- - \bigcup D_k$ is a Swiss cheese intersecting each line $L_1, L_2, \cdots$ in a set of zero linear measure.

4. REMARKS

Whether the statement of Theorem 1 remains true if, in the definition of "areally disconnected set", the hypothesis on a boundary curve of a regular open set A that it be piecewise C^2 is weakened to the requirement that it be only piecewise C^1, or possibly just rectifiable, is not known. The issue here, as far as concerns the proof of Theorem 1 given above, is whether analytic negligibility of C holds under these relaxed conditions. It may be noted that Vitushkin [15] has extended the collection of analytically negligible sets to include "Liapunov curves" (*cf.* Zalcman [16, p. 115]), and that Davie [2, Section 4] has extended this latter set somewhat further.

REFERENCES

1. K. F. Clancey and C. R. Putnam, *The local spectral behavior of completely subnormal operators.* Trans. Amer. Math. Soc. 163 (1972), 239-244.

2. A. M. Davie, *Analytic capacity and approximation problems.* Trans. Amer. Math. Soc. 171 (1972), 409-444.

3. A. M. Davie and B. K. Øksendal, *Rational approximation on the union of sets.* Proc. Amer. Math. Soc. 29 (1971), 581-584.

4. T. W. Gamelin, *Uniform algebras.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1969.

5. P. R. Halmos, *A Hilbert space problem book.* D. Van Nostrand Co., Inc., Princeton, N.J., 1967.

6. F. Hartogs and A. Rosenthal, *Über Folgen analytischer Funktionen.* Math. Ann. 104 (1931), 606-610.

7. R. G. Lautzenheiser, *Spectral sets, reducing subspaces, and function algebras.* Thesis, Indiana University, 1973.

8. C. R. Putnam, *An inequality for the area of hyponormal spectra.* Math. Z. 116 (1970), 323-330.

9. ———, *Peak sets and subnormal operators.* Illinois J. Math., to appear.

10. F. and M. Riesz, *Über die Randwerte einer analytischen Funktion.* Quatrième Congrès des Math. Scand., Stockholm (1916), 27-44.

11. A. Roth, *Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen.* Comment. Math. Helv. 11 (1938), 77-125.

12. W. Rudin, *Real and complex analysis.* Second Edition. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1974.

13. B. Sz.-Nagy and C. Foiaş, *Sur les contractions de l'espace de Hilbert. IV.* Acta Sci. Math. Szeged 21 (1960), 251-259.

14. J. L. Walsh, *Über die Entwicklung einer analytischen Funktion nach Polynomen.* Math. Ann. 96 (1926), 430-436.

15. A. G. Vitushkin, *The analytic capacity of sets in problems of approximation theory.* Uspehi Mat. Nauk 22 (1967), no. 6, 141-199. Russian Math. Surveys 22 (1967), 139-200.

16. L. Zalcman, *Analytic capacity and rational approximation.* Lecture Notes in Mathematics, No. 50. Springer-Verlag, Berlin-New York, 1968.

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