COBORDISM CLASSES REPRESENTED BY FIBERINGS WITH FIBER $\mathbb{R}P(2k+1)$

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1. INTRODUCTION

Let k be a nonnegative integer. Let $\eta_{n-k}(BO(k+1))$ be the unoriented cobordism group of real $(k+1)$-plane bundles over closed smooth $(n-k)$-dimensional manifolds. Let $\sigma_n^k\colon \eta_{n-k}(BO(k+1)) \to \eta_n$ be the homomorphism defined by assigning to the $(k+1)$-plane bundle ξ over M^{n-k} the cobordism class of the total space $\mathbb{R}P(\xi)$ of the associated projective space bundle. Many problems in cobordism theory can be reduced or related to the computation of this homomorphism. For instance, Stong [6; 8.4] proved that the image of σ_n^k is the set of cobordism classes in η_n which are represented by the total space of a fibering $\mathbb{R}P(k) \xrightarrow{i} M^n \xrightarrow{\pi} B^{n-k}$ which is totally nonhomologous to zero. Another example of the usefulness of σ_n^k was described in [1]: Let J_n^k be the set of cobordism classes in η_n which are represented by a manifold admitting an involution whose fixed point set is $(n-k)$-dimensional. Then the image of σ_n^k contains J_n^k, which in turn contains the image of σ_n^{2k-1}.

The main results of this paper are the following:

PROPOSITION 2.3. *The image of* σ_n^3 *equals the set of classes in* η_n *which are represented by a fibering with fiber* $\mathbb{R}P(3)$, *and is the set of classes* α *in* η_n *with* $w_1^j w_{n-j}(\alpha) = 0$ *for all* j, $0 \leq j \leq n$.

PROPOSITION 4.4. *The image of* σ_n^5 *equals* J_n^3, *and is the set of classes* α *in* η_n *with* $w_1^j w_{n-j}(\alpha) = w_1^{i-5} w_{n-i} s_5(\alpha) = 0$ *for all* j *and* i, $0 \leq j \leq n$, $5 \leq i \leq n$.

2. THE IMAGE OF σ_n^3

PROPOSITION 2.0. *Let* $f\colon M^n \to B^b$ *be a smooth map and let* $F = f^{-1}(p)$ *be the inverse image of a regular value of* f. *Let* $i\colon F \to M$ *be the inclusion. Then* $i_*[F] = f^*[B] \cap [M]$.

Proof. By examining tubular neighborhoods of F and p, we see by naturality that $f^*[B]$ is equal to what Milnor and Stasheff call the dual cohomology class to F in M [4, page 120]. The proposition then follows from [4; Problem 11-c].

COROLLARY. *If* $F^f \xrightarrow{i} M \xrightarrow{\pi} B$ *is a smooth fibering, then for any class* $x \in H^f(M; \mathbb{Z}_2)$ *the numbers* $\langle i^*(x), [F]\rangle$ *and* $\langle x \cup \pi^*[B], [M]\rangle$ *are equal.*

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The above corollary is due to D. O'Reilly [5]. Proposition 2.0 may be well-known and was pointed out to me by C. A. McGibbon. Finally, I am indebted to R. E. Stong for the following proposition.

PROPOSITION 2.1. *If* $\alpha \in \eta_n$ *is represented by a fibering with fiber* $\mathbb{R}P(2k+1)$, *then* $w_1^j w_{n-j}(\alpha) = 0$ *for all* j, $0 \leq j \leq n$.

Proof. Suppose α is represented by the fibering $\mathbb{R}P(2k+1) \xrightarrow{i} M^n \xrightarrow{\pi} B^{n-2k-1}$. The tangent bundle τM of M splits as a Whitney sum $\pi^* \tau B \oplus \theta$, where θ is the $(2k+1)$-plane bundle tangent to the fibers. Thus for $0 \leq j \leq n$,

$$w_{n-j}(M) = \sum_{p+q=n-j} w_p(\theta)\pi^*(w_q(B)) .$$

Since i is an embedding with trivial normal bundle, $i^*(w_1(M)) = w_1(\mathbb{R}P(2k+1)) = 0$, and from the Serre spectral sequence, $w_1(M) \in \text{image } \pi^*$. Because B is $(n-2k-1)$-dimensional, θ is $(2k+1)$-dimensional, and $w_1^j(M) \in \text{image } \pi^*$, we conclude that $w_1^j w_{n-j}(M) = w_{2k+1}(\theta) w_1^j(M) \pi^*(w_{n-j-2k-1}(B))$. Since

$$\langle i^*(w_{2k+1}(\theta)), [\mathbb{R}P(2k+1)] \rangle = \langle w_{2k+1}(\mathbb{R}P(2k+1)), [\mathbb{R}P(2k+1)] \rangle = 0,$$

the proposition now follows from O'Reilly's theorem.

By Proposition 2.1, the set of classes α in η_n with $w_1^j w_{n-j}(\alpha) = 0$, $0 \leq j \leq n$, contains the image of σ_n^3. We shall now show that the reverse statement is also true.

Let (n_1, n_2, n_3, n_4) be a partition of $n-3$. Let

$$\pi\colon \mathbb{R}P(n_1, \cdots, n_4) \to \mathbb{R}P(n_1) \times \cdots \times \mathbb{R}P(n_4)$$

be the projective space bundle associated to $\lambda_1 \oplus \cdots \oplus \lambda_4 \to \mathbb{R}P(n_1) \times \cdots \times \mathbb{R}P(n_4)$, where λ_i is the pullback of the canonical line bundle over the ith factor. Since by [2] $\eta_*(BO(4))$ is the ideal over η_* generated by the bundles

$$\lambda_1 \oplus \cdots \oplus \lambda_4 \to \mathbb{R}P(n_1) \times \cdots \times \mathbb{R}P(n_4),$$

the image of σ_*^3 is the ideal generated by the Stong manifolds $\mathbb{R}P(n_1, \cdots, n_4)$. In [6; 8.3] Stong exhibited for each integer n, $(n \neq 2^s, 2^s - 1)$ a partition $(n_1, \cdots, n_4)$ of $n-3$ such that the class x_n of $\mathbb{R}P(n_1, \cdots, n_4)$ is indecomposable in η_*. For $n = 2^s$, $s > 2$, Stong also exhibited a partition $(n_1, \cdots, n_4)$ of $n-3$ such that the class y_n of $\mathbb{R}P(n_1, \cdots, n_4)$ has $s_{2^{s-1}, 2^{s-1}}(y_n) = 1 \text{ Mod } 2$.

Let x_2 be the class of $\mathbb{R}P(2)$, and let x_{2^s} be the class of $\mathbb{R}P(2^s) \cup \mathbb{R}P(2)^{2^{s-1}}$ for $s > 1$. It is a classical result of Thom [7] that η_* is a $\mathbb{Z}_2$-polynomial algebra on the classes x_n.

PROPOSITION 2.2. *The numbers* $w_1^j w_{n-j}$, $0 \leq j \leq n$, *distinguish the classes* x_2^i *and* $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k$ *where* $n = 2i = 2^{r_1} + \cdots + 2^{r_t} + 2k$ *for* $i, k \geq 0$ *and* $r_1 > \cdots > r_t \geq 2$.

Proof. Suppose $n = 2^{r_1} + \cdots + 2^{r_t} + 2k$ for $k \geq 0$ and $r_1 > \cdots > r_t \geq 2$. Let

ω_k denote the ordered tuple $(2^{r_1}, \cdots, 2^{r_t}, 2k)$ and x_{ω_k} denote the product $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k$. Since $w_n(x_2^i) = 1$ Mod 2 while $w_n(x_{\omega_k}) = 0$ Mod 2 for all k, and since $w_1^n(x_{\omega_k}) = 1$ Mod 2 for $k < 2$ while $w_1^n(x_{\omega_k}) = 0$ Mod 2 for $k \geq 2$, it suffices to prove the numbers $w_1^j w_{n-j}$ distinguish the classes x_{ω_k} for $2 \leq k < n/2$. Define $\omega_p < \omega_q$ if $p > q$. Suppose $\omega_p < \omega_q$ and

$$\omega_p = (2^{p_1}, \cdots, 2^{p_t}, 2p) \quad \text{and} \quad \omega_q = (2^{q_1}, \cdots, 2^{q_u}, 2q).$$

Let a be the integer such that $p_i = q_i$ for $i < a$ and $q_a > p_a$. Let $j = 2^{q_a}$. Then $w_1^j w_{n-j}(x_{\omega_p}) = 0$ Mod 2 while $w_1^j w_{n-j}(x_{\omega_q}) = 1$ Mod 2. Furthermore, if $\omega_g < \omega_p$, then $w_1^j w_{n-j}(x_{\omega_g}) = 0$ Mod 2. This completes the proof.

Since the numbers $w_1^j w_{n-j}$, $0 \leq j \leq n$, all vanish on the products $x_{2^s-1}^2 x_2^i$ and $x_{2^s-1}^2 x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k$, $s > 2$, we see that the set of classes in η_* with $w_1^j w_{n-j} = 0$ contains all products of generators for η_* except those listed in Proposition 2.2. Therefore if $n = 2^s$, $s > 2$, then the class $y_n = x_{2^s-1}^2$ modulo the ideal generated by lower dimensional classes in the image of σ_*^3. Hence the image of σ_*^3 contains all products of generators for η_* except those listed in Proposition 2.2. Therefore, the image of σ_*^3 is the ideal generated by the x_n, $n \neq 2^s$, and the classes y_{2^s}, $s > 2$; and, we have proven

PROPOSITION 2.3. *The image of* σ_n^3 *equals the set of classes in* η_n *which are represented by a fibering with fiber* $\mathbb{R}P(3)$, *and is the set of classes* α *in* η_n *with* $w_1^j w_{n-j}(\alpha) = 0$ *for all* j, $0 \leq j \leq n$.

We should point out that Proposition 8.3 of [6] is false, and that Proposition 2.3 gives the correct version.

3. NECESSARY CONDITIONS FOR CLASSES BELONGING TO J_n^3

Since the image of σ_n^3 contains J_n^3, if α is in J_n^3 then $w_1^j w_{n-j}(\alpha) = 0$ for all j, $0 \leq j \leq n$. In this section we shall show that $w_1^{i-5} w_{n-i} s_5(\alpha) = 0$ for all i, $5 \leq i \leq n$, as well. Suppose α is represented by the manifold M^n with involution T whose fixed set F^{n-3} has codimension 3. Let $\nu^3 \to F^{n-3}$ be the normal bundle. By [3; 24.2], M^n is cobordant to $\mathbb{R}P(\nu^3 \oplus \mathbb{R})$, where $\mathbb{R}P(\nu^3 \oplus \mathbb{R})$ is the total space of the projective space bundle associated to $\nu^3 \oplus \mathbb{R} \to F^{n-3}$. Therefore, it suffices to show $w_1^{i-5} w_{n-i} s_5(\mathbb{R}P(\nu^3 \oplus \mathbb{R})) = 0$ for all i, $5 \leq i \leq n$.

Following [3], $H^*(\mathbb{R}P(\nu^3 \oplus \mathbb{R}))$ is a free $H^*(F^{n-3}; \mathbb{Z}_2)$-module via the map p^* indiced by the projection on classes 1, c, c^2, c^3 where c is the characteristic class of the canonical line bundle over $\mathbb{R}P(\nu^3 \oplus \mathbb{R})$. Multiplication in

$$H^*(\mathbb{R}P(\nu^3 \oplus \mathbb{R}); \mathbb{Z}_2)$$

is subject to the relation $c^4 = c^3 p^*(v_1) + c^2 p^*(v_2) + cp^*(v_3)$, where v_i, $1 \leq i \leq 3$, denotes the ith Whitney class of ν^3. Furthermore,

$$w(\mathbb{R}P(\nu^3 \oplus \mathbb{R}P)) = p^*(w(F^{n-3}))(1 + p^*(v_1) + p^*(v_1)c + p^*(v_2) + p^*(v_3) + p^*(v_1)c^2).$$

If w_j denotes the jth Whitney class of F^{n-3}, then $w_1(\mathbb{R}P(\nu \oplus \mathbb{R})) = p^*(w_1 + v_1)$ and

$$w_{n-i}(\mathbb{R}P(\nu \oplus \mathbb{R})) = p^*(w_{n-i-3}v_3 + w_{n-i-2}v_2 + w_{n-i-1}v_1 + w_{n-i}) + p^*(w_{n-i-3}v_1)c^2 + p^*(w_{n-i-2}v_1)c.$$

From [2],

$$s_5(\mathbb{R}P(\nu \oplus \mathbb{R})) = p^*(s_5(F^{n-3}) + s_5(\nu)) + p^*(s_4(\nu))c + p^*(s_1(\nu))c^4.$$

Because F is (n - 3)-dimensional, this means

$$\begin{aligned} w_1^{i-5}w_{n-i}s_5(\mathbb{R}P(\nu \oplus \mathbb{R})) &= p^*((w_1 + v_1)^{i-5}w_{n-i-3}v_1 s_4(\nu))c^3 \\ &+ p^*((w_1 + v_1)^{i-5}(w_{n-i-3}v_3 + w_{n-i-2}v_2 + w_{n-i-1}v_1 + w_{n-i})s_1(\nu))c^4 \\ &+ p^*((w_1 + v_1)^{i-5}w_{n-i-3}v_1 s_1(\nu))c^6 + p^*((w_1 + v_1)^{i-5}w_{n-i-2}v_1 s_1(\nu))c^5. \end{aligned}$$

Because $s_{2^r}(\nu) = v_1^{2^r}$ and $c^4 = c^3p^*(v_1) + c^2p^*(v_2) + cp^*(v_3)$,

$$w_1^{i-5}w_{n-i}s_5(\mathbb{R}P(\nu \oplus \mathbb{R}))$$

reduces even further to

$$p^*((w_1 + v_1)^{i-5}(w_{n-i}v_1^2 + w_{n-i-1}v_1^3 + w_{n-i-2}v_1^4))c^3.$$

Therefore, the number

$$\begin{aligned} &\langle w_1^{i-5}w_{n-i}s_5(\mathbb{R}P(\nu \oplus \mathbb{R})), [\mathbb{R}P(\nu \oplus \mathbb{R})]\rangle \\ &\qquad = \langle (w_1 + v_1)^{i-5}(w_{n-i}v_1^2 + w_{n-i-1}v_1^3 + w_{n-i-2}v_1^4), [F^{n-3}]\rangle. \end{aligned}$$

We shall show that this number is zero by considering the canonical line bundle $\lambda \to \mathbb{R}P(\nu^3)$.

From [3], $\lambda \to \mathbb{R}P(\nu)$ bords as an element of $\eta_{n-1}(BO(1))$. Hence all the characteristic numbers of $\lambda \to \mathbb{R}P(\nu)$ vanish. If e denotes the characteristic class of λ, and $q\colon \mathbb{R}P(\nu) \to F^{n-3}$ denotes the projective space bundle associated to $\nu^3 \to F^{n-3}$, then we have the following facts [3]: $H^*(\mathbb{R}P(\nu); \mathbb{Z}_2)$ is a free $H^*(F^{n-3}; \mathbb{Z}_2)$-module via q^* on the classes 1, e, e^2 subject to the relation $e^3 = e^2q^*(v_1) + eq^*(v_2) + q^*(v_3)$.

$$w(\mathbb{R}P(\nu)) = q^*(w(F^{n-3}))(1 + q^*(v_1) + e + q^*(v_2) + e^2).$$

Therefore, $w_1(\mathbb{R}P(\nu)) + e = q^*(w_1 + v_1)$ and

$$w_{n-j}(\mathbb{R}P(\nu)) = q^*(w_{n-j} + w_{n-j-1}v_1 + w_{n-j-2}v_2) + q^*(w_{n-j-1})e + q^*(w_{n-j-2})e^2.$$

Thus for each i, $5 \leq i \leq n$,

$$(w_1(\mathbb{R}P(\nu)) + e)^{i-5}(w_{n-i+4}(\mathbb{R}P(\nu)) + w_{n-i+3}(\mathbb{R}P(\nu))e + w_{n-i+1}(\mathbb{R}P(\nu))e^3$$
$$+ w_{n-i}(\mathbb{R}P(\nu))e^4) = q^*((w_1 + v_1)^{i-5}(w_{n-i} v_1^2 + w_{n-i-1} v_1^3 + w_{n-i-2} v_1^4))e^2 .$$

Therefore,

$$\left\langle (w_1 + v_1)^{i-5} (w_{n-i} v_1^2 + w_{n-i-1} v_1^3 + w_{n-i-2} v_1^4),\ [F^{n-3}] \right\rangle$$
$$= \left\langle (w_1(\mathbb{R}P(\nu)) + e)^{i-5} (w_{n-i+4} (\mathbb{R}P(\nu)) + w_{n-i+3} (\mathbb{R}P(\nu))e + w_{n-i+1} (\mathbb{R}P(\nu))e^3 \right.$$
$$\left. + w_{n-i} (\mathbb{R}P(\nu))e^4),\ [\mathbb{R}P(\nu)] \right\rangle = 0 .$$

This proves

PROPOSITION 3.1. J_n^3 *is contained in the set of classes* α *in* η_n *with* $w_1^j w_{n-j}(\alpha) = w_1^{i-5} w_{n-i} s_5(\alpha) = 0$ *for all* j *and* i, $0 \leq j \leq n$, $5 \leq i \leq n$.

4. A GENERATING SET FOR J_*^3

In this section we shall show that J_*^3 contains certain classes which comprise a generating set for the set of classes α in η_n with $w_i^j w_{n-j}(\alpha) = w_1^{i-5} w_{n-i} s_5(\alpha) = 0$ for all j and i, $0 \leq j \leq n$, $5 \leq i \leq n$.

In analogy with section 2, suppose $(n_1, \cdots, n_6)$ is a partition of n - 5. Then the class of the Stong manifold $\mathbb{R}P(n_1, \cdots, n_6)$ lies in the image of σ_n^5, which is in turn contained in J_n^3. A useful property [6; 3.4] of these manifolds is that the class of $\mathbb{R}P(n_1, \cdots, n_6)$ is indecomposable in η_* if and only if $\binom{n-1}{n_1} + \cdots + \binom{n-1}{n_6}$ is odd.

PROPOSITION 4.1. *For each integer* $n \geq 9$ $(n \neq 2^s, 2^s - 1)$, J_n^3 *contains an indecomposable class* x_n.

Proof. By the discussion preceding this Proposition, it suffices to exhibit a partition $(n_1, \cdots, n_6)$ of n - 5 for each integer $n \geq 9$, $(n \neq 2^s, 2^s - 1)$ such that $\mathbb{R}P(n_1, \cdots, n_6)$ is indecomposable. If $\binom{n-1}{n-5} = 0$ Mod 2, then $\mathbb{R}P(n - 5, \underbrace{0, \cdots, 0}_{5})$ is as required. Suppose $\binom{n-1}{n-5} = 1$ Mod 2. Since $\binom{n-1}{n-5} = \binom{n-1}{4}$, the dyadic expansion of n - 1, say $2^{r_1} + \cdots + 2^{r_t}$, $r_1 > \cdots > r_t \geq 0$, contains 2^2 as a term. Since $n \geq 9$, if $r_t = 2$ or if $r_t = 0$ and $r_{t-1} = 2$, then $\mathbb{R}P(n - 7, 1, 1, 0, 0, 0)$ is as required. If $r_t = 1$, since $n \geq 9$, $n \neq 2^s$, $2^s - 1$, there exists an integer i, $1 \leq i \leq t - 2$, such that $r_i > r_{i+1} + 1$. Then

$$\mathbb{R}P(2^{r_1} + \cdots + 2^{r_i} - 2,\ 2^{r_{i+1}} + \cdots + 2^{r_t} - 6,\ 4, 0, 0, 0)$$

is as required.

Because $\mathbb{R}P(n_1, \cdots, n_6)$ is the total space of the projective space bundle associated to $\lambda_1 \oplus \cdots \oplus \lambda_6 \to \mathbb{R}P(n_1) \times \cdots \times \mathbb{R}P(n_6)$ we can explicitly describe its cohomology and Whitney class [3]: $H^*(\mathbb{R}P(n_1, \cdots, n_6); \mathbf{Z}_2)$ is a free

$$H^*(\mathbb{R}P(n_1) \times \cdots \times \mathbb{R}P(n_6); \mathbb{Z}_2)\text{-module}$$

via the map p^* induced by the projection on the classes $1, c, \cdots, c^5$ where c is the characteristic class of the canonical line bundle over $\mathbb{R}P(n_1, \cdots, n_6)$ and satisfies the relation $c^6 = c^5 p^*(v_1) + c^4 p^*(c_2) + \cdots + p^*(v_6)$, where v_i, $1 \le i \le 6$, is the ith Whitney class of $\lambda_1 \oplus \cdots \oplus \lambda_6$.

$$w(\mathbb{R}P(n_1, \cdots, n_6))$$
$$= p^*(w(\mathbb{R}P(n_1) \times \cdots \times \mathbb{R}P(n_6)))((1+c)^6 + (1+c)^5 p^*(v_1) + \cdots + p^*(v_6)) .$$

PROPOSITION 4.2. *If* $n = 2^s$, $s > 2$, *then* J_n^3 *contains a class* y_n *with* $s_{2^{s-1}, 2^{s-1}}(y_n) = 1$ Mod 2.

Proof. If suffices to exhibit a partition $(n_1, \cdots, n_6)$ of $n - 5$ for each $n = 2^s$, $s > 2$, such that

$$w_2^{2^{s-1}}(\mathbb{R}P(n_1, \cdots, n_6)) = s_{2^{s-1}, 2^{s-1}}(\mathbb{R}P(n_1, \cdots, n_6)) = 1 \quad \text{Mod } 2 .$$

From the paragraph preceding this proposition, we conclude that if $n = 8$, $\mathbb{R}P(3, \underbrace{0, \cdots, 0}_{5})$ is as required, while if $n = 2^s$, $s > 3$,

$$\mathbb{R}P(2^{s-2} - 2,\ 2^{s-2} - 1,\ 2^{s-2} - 1,\ 2^{s-2} - 1,\ 0,\ 0)$$

is as required.

Let x_2 be the cobordism class of $\mathbb{R}P(2)$; let x_5 be the class of $\mathbb{R}P(2, 0, 0, 0)$; let x_6 be the class of $\mathbb{R}P(6) \cup \mathbb{R}P(2) \times \mathbb{R}P(4)$; if $n = 2^s$, $s > 1$, let x_n be the class of $\mathbb{R}P(2^s) \cup \mathbb{R}P(2)^{2^{s-1}}$. Then η_* is a $\mathbb{Z}_2$-polynomial algebra on the classes x_n given here and in Proposition 4.1.

A direct computation shows that x_5^2 is the class of $\mathbb{R}P(2, 2, 1, 0, 0, 0)$, that $x_5 x_6$ is the class of $\mathbb{R}P(3, 2, 1, 0, 0, 0) \cup \mathbb{R}P(2) \times \mathbb{R}P(2, 1, 1, 0, 0, 0)$, and that x_6^2 is the class of $\mathbb{R}P(3, 2, 2, 0, 0, 0) \cup \mathbb{R}P(2) \times \mathbb{R}P(2, 2, 1, 0, 0, 0)$.

By Proposition 2.3, the image of σ_*^3 is generated by the x_n, $n \ne 2^s$, given here and the y_{2^s}, $s > 2$, given in Proposition 4.2. We claim that J_*^3, which is contained in the image of σ_*^3, contains all products of generators for η_* contained in the image of σ_*^3 except those of the form $x_2^j x_5$, $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_5$, $x_2^j x_6$, and $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_6$. This claim follows from:

PROPOSITION 4.3. *The numbers* $w_1^{i-5} w_{n-i} s_5$, $5 \le i \le n$, *distinguish the classes* $x_2^j x_5$, $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_5$, *and the classes* $x_2^j x_6$, $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_6$, *where* $j, k \ge 0$ *and* $r_1 > \cdots > r_t \ge 2$.

Proof. Since $x_2^j x_5$ and $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_5$ are odd-dimensional, while $x_2^j x_6$ and $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_6$ are even-dimensional, the former terms cannot appear with any of the latter. Thus we may divide the proof into separate arguments.

Suppose $n = 2^{r_1} + \cdots + 2^{r_t} + 2k + 5$ for $k \geq 0$ and $r_1 > \cdots > r_t \geq 2$. Let ω_k denote the ordered tuple $(2^{r_1}, \cdots, 2^{r_t}, 2k+5)$ and x_{ω_k} denote the product $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_5$. Since $w_{n-5}\, s_5(x_2^j x_5) = 1$ Mod 2 for $n = 2j + 5$, while $w_{n-5}\, s_5(x_{\omega_k}) = 0$ Mod 2 for all k, and since $w_1^{n-5}\, s_5(x_{\omega_k}) = 1$ Mod 2 for $k < 2$ while $w_1^{n-5}\, s_5(x_{\omega_k}) = 0$ Mod 2 for $k \geq 2$, it suffices to prove the numbers $w_1^{i-5}\, w_{n-i}\, s_5$ distinguish the classes x_{ω_k} for $2 \leq k < (n-5)/2$. Define $\omega_p < \omega_q$ if $p > q$. Suppose $\omega_p < \omega_q$ and

$$\omega_p = (2^{p_1}, \cdots, 2^{p_t}, 2p+5) \quad \text{and} \quad \omega_q = (2^{q_1}, \cdots, 2^{q_u}, 2q+5).$$

Let a be the integer such that $p_i = q_i$ for $i < a$ and $q_a > p_a$. Let $j = 2^{q_a}$. Then $w_1^j\, w_{n-j-5}\, s_5(x_{\omega_p}) = 0$ Mod 2 while $w_1^j\, w_{n-j-5}\, s_5(x_{\omega_q}) = 1$ Mod 2. Furthermore, if $\omega_g < \omega_p$, then $w_1^j\, w_{n-j-5}\, s_5(x_{\omega_g}) = 0$ Mod 2. This proves that the numbers $w_1^{i-5}\, w_{n-i}\, s_5$, $5 \leq i \leq n$, distinguish the classes $x_2^j x_5$ and $x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_5$.

To complete the proof, we remark that the argument for terms ending in x_6 may be obtained simply by substituting x_6 for x_5 and letting

$$\omega_k = (2^{r_1}, \cdots, 2^{r_t}, 2k+6)$$

in the above paragraph.

Let us summarize our arguments to this point: From Section 3, J_n^3 is contained in the set of classes α in η_n with $w_1^j\, w_{n-j}(\alpha) = w_1^{i-5}\, w_{n-i}\, s_5(\alpha) = 0$ for all j and i, $0 \leq j \leq n$, $5 \leq i \leq n$. By Proposition 4.1, J_n^3 contains the indecomposable class x_n for each integer $n \geq 9$, $n \neq 2^s$, $2^s - 1$. Furthermore, J_*^3 contains x_5^2, $x_5 x_6$, and x_6^2. By Proposition 4.2, J_n^3 contains a class y_n, $n = 2^s$, $s > 2$, such that $s_{2^{s-1}, 2^{s-1}}(y_n) = 1$ Mod 2. Since the numbers $w_1^{i-5}\, w_{n-i}\, s_5$, $5 \leq i \leq n$, all vanish on the products $x_{2^r}^2 x_2^j x_5$, $x_{2^r}^2 x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_5$, $x_{2^r}^2 x_2^j x_6$, and

$$x_{2^r}^2 x_{2^{r_1}} \cdots x_{2^{r_t}} x_2^k x_6 \qquad \text{for } r > 1,$$

we conclude by Proposition 4.3 that the numbers $w_1^{i-5}\, w_{n-i}\, s_5$, $5 \leq i \leq n$, vanish on all products of generators for η_* contained in the image of σ_*^3 except those listed in Proposition 4.3. Thus $y_{2^s} = x_{2^{s-1}}^2$, $s > 2$, modulo the ideal generated by lower dimensional classes of J_*^3. Therefore, J_*^3 contains all products of generators for η_* contained in the image of σ_*^3 except those listed in Proposition 4.3. Since the generators for J_*^3 we have constructed all lie in the image of σ_*^5, we have proved

PROPOSITION 4.4. *The image of* σ_n^5 *equals* J_n^3, *and is the set of classes* α *in* η_n *with* $w_1^j\, w_{n-j}(\alpha) = w_1^{i-5}\, w_{n-i}\, s_5(\alpha) = 0$ *for all* j *and* i, $0 \leq j \leq n$, $5 \leq i \leq n$.

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