ON RIESZ TRANSFORMS OF BOUNDED FUNCTIONS OF COMPACT SUPPORT

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1. Let K be a compact set of $\mathbb{R}^n$ such that $m(K) > 0$, where m is n-dimensional Lebesgue measure. Let $L^\infty(K)$ denote the set of all functions in $L^\infty(\mathbb{R}^n)$ which vanish almost everywhere on $\mathbb{R}^n \setminus K$. We will be concerned with the set $H^\infty(K)$ of those functions in $L^\infty(K)$ which have bounded Riesz transforms. More precisely, a function $h \in L^\infty(K)$ is in $H^\infty(K)$ if and only if all Riesz transforms

$$R_j h(x) = \text{P.V. } c_n \int \frac{(x_j - t_j)}{|x - t|^{n+1}} h(t)\,dt, \qquad j = 1, 2, \cdots, n,$$

where c_n is normalizing constant depending only on n, belong to $L^\infty(\mathbb{R}^n)$. It follows from a classical result that $\|R_j h\|_p \leq A_p \|h\|_p$ if $1 < p < \infty$. (See Stein [9].) When $p = \infty$, $R_j h$ does not necessarily belong to $L^\infty(\mathbb{R}^n)$. In fact, it is relatively simple to show that there exists a function $h \in L^\infty(K)$ such that $R_j h \notin L^\infty(\mathbb{R}^n)$ for all $j = 1, 2, \cdots, n$. The main purpose of this paper is to investigate whether or not $H^\infty(K)$ is always *nontrivial; i.e.*, $H^\infty(K) \neq \{0\}$. We remark that $H^\infty(K)$ is a Banach space under the norm $\|h\| = \|h\|_\infty + \sum_{j=1}^n \|R_j h\|_\infty$. Related to $H^\infty(K)$ is the set $\mathscr{H}(K)$ of bounded harmonic functions defined on $\mathbb{R}^{n+1} \setminus K$ and satisfying a Lipschitz condition. If $\mathscr{H}(K)$ consists only of the constants, the set K is called *removable* for harmonic functions satisfying a Lipschitz condition. It turns out that K is removable if and only if $H^\infty(K)$ is trivial (see Theorem 1). We should mention here the related work of Harvey and Polking [6], where they have found sufficient conditions on removable sets for solutions of linear partial differential equations. We remark that the well-known result that $m(K) = 0$ implies K is removable for harmonic functions satisfying a Lipschitz condition, can also be derived from their Theorem 4.3(b).

The problem of removable singularities of harmonic functions satisfying a Lipschitz condition of order α, $0 < \alpha < 1$, has been completely solved by Carleson (see [3, Section VII, Theorem 2]). He proved that K is removable if and only if the $(n - 2 + \alpha)$-dimensional Hausdorff measure $\Lambda_{n-2+\alpha}(K) = 0$.

THEOREM 1. *Let* K *be a compact set of* $\mathbb{R}^n$. *Then* $u \in \mathscr{H}(K)$ *if and only if there exists a function* $h \in H^\infty(K)$ *such that*

$$u(x, y) = \int \log\{(x - t)^2 + y^2\}\, h(t)\,dt + \text{Constant} \qquad \text{if } n = 1$$

and

$$u(x, y) = \int \frac{h(t)}{(|x - t|^2 + y^2)^{(n-1)/2}}\,dt + \text{Constant} \qquad \text{if } n > 1.$$

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Proof. (For $n > 1$). Suppose $u \in \mathscr{H}(K)$. Since u can be extended continuously across K, we find $u(x, y) = u(x, -y)$. It follows that $\frac{\partial u}{\partial y}(x, -y) = -\frac{\partial u}{\partial y}(x, -y)$ and that

$$\frac{\partial u}{\partial y}(x, y) = c_n \int \frac{y\, h(t)}{(|x - t|^2 + y^2)^{(n+1)/2}}\, dt \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^{n+1} \setminus K,$$

where $h(t) = \lim_{y \to 0} \frac{\partial u}{\partial y}(t, y)$ exists almost everywhere and vanishes on $\mathbb{R}^n \setminus K$. Therefore,

$$u(x, y) = \frac{c_n}{1 - n} \int \frac{h(t)}{(|x - t|^2 + y^2)^{(n-1)/2}}\, dt + \text{Constant}.$$

To complete the proof of the theorem, we observe that

$$\frac{\partial u}{\partial x_j}(x, y) = c_n \int \frac{(x_j - t_j)\, h(t)}{(|x - t|^2 + y^2)^{(n+1)/2}}\, dt = c_n \int \frac{y\, R_j h(y)}{(|x - t|^2 + y^2)^{(n+1)/2}}\, dt, \qquad y > 0.$$

(See [10, Chapter VI, Theorem 4.17].) This proves that $\frac{\partial u}{\partial x_j}$ is bounded if and only if $R_j h \in L^\infty(\mathbb{R}^n)$, and the theorem follows.

THEOREM 2. *If* $n = 1$, *then* $H^\infty(K)$ *is nontrivial for any compact set* K *with* $m(K) > 0$.

Proof. Suppose $m(K) > 0$, and let f be the Ahlfors function for K. (See [5, Chapter VIII].) It is well known that f is nonconstant. By applying Cauchy's integral formula, we find $f(z) = \int_K \frac{h(t)}{z - t}\, dt$, $z \notin K$, where

$$h(t) = \lim_{y \to 0} \frac{1}{2\pi i} \{f(t + iy) - f(t - iy)\},$$

which exists almost everywhere and vanishes on $\mathbb{R} \setminus K$. Furthermore, since $f(z) = \overline{f(\bar{z})}$, h is a real-valued function. Therefore, if u is the real part of f, then

$$u(x, y) = \int \frac{(x - t)\, h(t)}{(x - t)^2 + y^2}\, dt = \int \frac{y H h(t)}{(x - t)^2 + y^2}\, dt, \qquad y > 0,$$

where H denotes the Hilbert transform. Since u is bounded, this implies that $Hh \in L^\infty(\mathbb{R})$, and the theorem follows.

2. From now on, we may assume that $n > 1$. We shall be involved with the Riesz capacities. We give here some notations for these capacities and refer to Landkof [8] for results. The *Riesz potential of order* α of μ is denoted by u_α^μ, where $u_\alpha^\mu(x) = \int \frac{d\mu(t)}{|x - t|^\alpha}$. If E is a Borel subset of $\mathbb{R}^n$, the *Riesz capacity of order* α of E is defined by the relation $C_\alpha(E) = \sup \mu(E)$, where the supremum is taken over all positive measures μ with support $S_\mu \subset E$ and potential $u_\alpha^\mu \leq 1$. We shall consider the capacitary condition

$$(*) \qquad C_\alpha(B \setminus K) < C_\alpha(B),$$

where B is some ball containing K. Whether B is open or closed does not matter; however, for convenience of the argument, we let B be open. Later on we will see that (*) depends only on the degree of density of K and is independent of the choice of B. The significance of this condition in the application of the theory of capacity can be found in several papers; *e.g.*, [1], [4], [7], and [2]. Notice that (*) does not hold if $0 < \alpha \leq n - 2$ or if $n - 2 < \alpha < n$ and $m(K) = 0$. The first case follows from the fact that the equilibrium measure of any compact set concentrates on its outer boundary. (See [8, p. 162].) The second case follows from the fact that the equilibrium measure of B is absolutely continuous with respect to Lebesgue measure. (See [8, Appendix].) Later on we will see that there exists a compact set K with $m(K) > 0$ but $C_\alpha(B \setminus K) = C_\alpha(B)$ for all $\alpha \in (n - 2, n)$.

THEOREM 3. *Let* K *be a compact set and* B *be an open ball containing* K *such that the condition* (*) *holds for some* $\alpha \in (n - 2, n)$. *Then* $H^\infty(K)$ *is nontrivial.*

Proof. We may assume without loss of generality that the interior $\overset{\circ}{K}$ is empty and B is the unit ball. Let μ be the equilibrium measure of $B \setminus K$, $u = u_\alpha^\mu$, and

$$h(t) = \begin{cases} 1 - u(t) & \text{if } t \in B, \\ 0 & \text{otherwise}. \end{cases}$$

By hypothesis, $h \neq 0$ on a set of positive measure contained in K. Since $u = 1$ on $B \setminus K$, it follows that $h \in L^\infty(K)$. We will prove that $R_j h \in L^\infty(\mathbb{R}^n)$ for all $j = 1, 2, \cdots, n$. It is clearly enough to show that the functions

$$u_j(x, y) = \int \frac{(x_j - t_j)\, h(t)}{(|x - t|^2 + y^2)^{(n+1)/2}}\, dt, \qquad j = 1, 2, \cdots, n,$$

are bounded on the set of (x, y) such that $x \in B' \setminus K$, where B' is some open ball containing K with radius less than 1. Because $u(x) = 1$, we can write

$$u_j(x, y) = \int_B d\mu(s) \int_B \left\{ \frac{1}{|x - s|^\alpha} - \frac{1}{|t - s|^\alpha} \right\} \frac{(x_j - t_j)}{(|x - t|^2 + y^2)^{(n+1)/2}}\, dt.$$

We will estimate $u_j(x, y)$ by dividing the inner integral into three parts over

$$E = \left\{ t \in B: |x - t| \leq \tfrac{1}{2}|x - s| \right\}, \quad F = \left\{ t \in B: \tfrac{1}{2}|x - s| \leq |x - t| \leq 2|x - s| \right\},$$

and

$$G = \{ t \in B: |x - t| \geq 2|x - s| \}.$$

Let $I(E)$, $I(F)$, $I(G)$ be the corresponding integrals, and let C be a certain absolute constant. Then

$$|I(E)| \le \int_B d\mu(s) \int_E \left| \left\{ \frac{1}{|x-s|^\alpha} - \frac{1}{|t-s|^\alpha} \right\} \right| \frac{dt}{|x-t|^n}$$

$$\le C \int_B d\mu(s) \int_E \frac{|x-t|^{\alpha/k}}{|x-s|^{\alpha+\alpha/k}} \frac{dt}{|x-t|^n}$$

$$\le C \int_B \frac{d\mu(s)}{|x-s|^{\alpha+\alpha/k}} \int_0^{\frac{1}{2}|x-s|} \frac{dt}{r^{1-\alpha/k}} \le C \int_B \frac{d\mu(s)}{|x-s|^\alpha} \le C,$$

where k is some fixed positive integer such that $\alpha/k \le 1$. Also,

$$|I(F)| \le \int_B d\mu(s) \int_F \frac{dt}{|x-t|^n} \int_B d\mu(s) \int_{\frac{1}{2}|x-s|}^{2|x-s|} \frac{dr}{r} \le C.$$

To estimate I(G), we write $I(G) = I'(G) - I''(G)$, where

$$I'(G) = \int_B \frac{d\mu(s)}{|x-s|^\alpha} \int_G \frac{(x_j - t_j)}{(|x-t|^2 + y^2)^{(n+1)/2}} dt$$

and

$$I''(G) = \int_B d\mu(s) \int_G \frac{1}{|t-s|^\alpha} \frac{(x_j - t_j)}{(|x-t|^2 + y^2)^{(n+1)/2}} dt.$$

Let $D = \{t \in \mathbb{R}^n : 2|x-s| \le |x-t| \le 2\}$. Since $\int_D \frac{(x_j - t_j)}{(|x-t|^2 + y^2)^{(n+1)/2}} dt = 0$,

$$I'(G) = -\int_B \frac{d\mu(s)}{|x-s|} \int_{D \setminus G} \frac{(x_j - t_j)}{(|x-t|^2 + y^2)^{(n+1)/2}} dt.$$

Therefore, $|I'(G)| \le C$, because $\operatorname{dist}(B', D \setminus G) > 0$. Finally,

$$|I''(G)| \le C \int_B d\mu(s) \int_G \frac{dt}{|x-t|^{n+\alpha}}$$

$$\le C \int_B d\mu(s) \int_{2|x-s|}^\infty \frac{dr}{r^{1+\alpha}} \le C \int_B \frac{d\mu(s)}{|x-s|^\alpha} \le C.$$

COROLLARY. *There exists a totally disconnected compact set* K *such that* $H^\infty(K)$ *is nontrivial.*

Proof. Consider a closed cube $Q \subset B$. Then $C_\alpha(B \setminus Q) < C_\alpha(B)$ for all $\alpha \in (n-2, n)$. Fix an $\alpha \in (n-1, n)$ and choose a sequence $\{a_k\}$ of positive numbers such that $C_\alpha(B \setminus Q) + \sum_{k=1}^\infty a_k < C_\alpha(B)$. Divide Q into 2^n cubes by n hyperplanes parallel to the faces of Q and passing through its center. Since for $n - 1 < \alpha < n$, the Riesz capacity of order α of any hyperplane (of dimension $n - 1$)

is 0, we can remove from Q a set V_1 symmetrically along these hyperplanes so that $C_\alpha(V_1) < a_1$ and $Q_1 = Q \setminus V_1$ is a union of 2^n disjoint closed cubes of the same size. Suppose at step $k \geq 1$ we have defined V_k and Q_k. We repeat the above division with each cube of Q_k and remove from Q_k a set V_{k+1} along the hyperplanes occurring at this step so that $C_\alpha(V_{k+1}) < a_{k+1}$ and $Q_k = Q_k \setminus V_{k+1}$ consists of $2^{n(k+1)}$ disjoint closed cubes of the same size. Thus $Q_1 \supset Q_2 \supset \cdots$. Let

$$K = \bigcap_{k=1}^{\infty} Q_k.$$

Then K is totally disconnected and $B \setminus K = (B \setminus Q) \cup \left(\bigcup_{k=1}^{\infty} V_k\right)$. Thus

$$C_\alpha(B \setminus K) \leq C_\alpha(B \setminus Q) + \sum_{k=1}^{\infty} C_\alpha(V_k) < C_\alpha(B \setminus Q) + \sum_{k=1}^{\infty} a_k < C_\alpha(B).$$

and the corollary follows.

3. The answer to the question of the nontriviality of $H^\infty(K)$ depends upon Theorem 3, where we have assumed that the condition (*) holds for some $\alpha \in (n - 2, n)$. In this section we will see that there exists a compact set of positive measure which does not satisfy (*) for all $\alpha \in (n - 2, n)$. Thus Theorem 3 gives only a partial answer. In general, the nontriviality of $H^\infty(K)$ is still unknown.

THEOREM 4. *Let* $\alpha \in (n - 2, n)$, *and let* E *be a compact set with* $m(E) > 0$. *Then for each* $\varepsilon > 0$, *there exists a compact set* $K \subset E$ *such that* $m(E \setminus K) < \varepsilon$ *and* $C_\alpha(B \setminus K) = C_\alpha(B)$, *where* B *is an open ball containing* K.

LEMMA. *Let* $\alpha \in (n - 2, n)$. *Then the following are equivalent.*

(i) $C_\alpha(B \setminus K) = C_\alpha(B)$;

(ii) $\limsup_{\delta \to 0} \dfrac{C_\alpha(Q(x, \delta) \setminus K)}{\delta^n} > 0$ *for almost all* $x \in K$, *where* $Q(x, \delta)$ *is the closed cube of center* x *and side* δ.

Proof. That (ii) implies (i) follows from Theorem 9 of [7]. To prove that (i) implies (ii), suppose there exists a sequence $\{\delta_k\}$ decreasing to 0 such that

$$\frac{C_\alpha(Q(x, \delta_k) \setminus K)}{\delta_k^n} \to 0 \quad \text{as } k \to \infty$$

for all $x \in F \subset K$ with $m(F) > 0$. By Egoroff's theorem, we may assume that the above convergence is uniform on F. For each $k = 1, 2, \cdots$, cover F with a finite collection $\{P_{k,j}\}$ of cubes intersecting F with sides equal to $\delta_k/2$ such that $\mathring{P}_{k,j} \cap \mathring{P}_{k,\ell} = \emptyset$ if $j \neq \ell$. Choose a point $x_{k,j} \in F \cap P_{k,j}$, and let $Q_{k,j} = Q(x_{k,j}, \delta_k)$. It is easy to see that $F \subset \bigcup_j Q_{k,j}$. Now choose $\varepsilon > 0$ and consider the equilibrium measure μ of $B \setminus K$. Since $\mu(Q_{k,j}) \leq C_\alpha(Q_{k,j} \setminus K)$, we obtain

$$\mu(F) \leq \sum_j C_\alpha(Q_{k,j} \setminus K) \leq \varepsilon \sum_j m(Q_{k,j}) \leq \varepsilon C$$

if k is sufficiently large. Hence $\mu(F) = 0$. This implies that $\mu \neq \nu$, where ν is the equilibrium measure of B. Therefore $C_\alpha(B \setminus K) < C_\alpha(B)$, which is a contradiction.

Proof of Theorem 4. Let $\{\delta_k\}$ be a sequence of positive numbers decreasing to 0, and let $\{\delta_k'\} = \{\delta_k^{n/\alpha}\}$. Cover E with a finite collection $\{Q_{k,j}\}$ of cubes intersecting E of sides equal to δ_k such that $\mathring{Q}_{k,j} \cap \mathring{Q}_{k,\ell} = \emptyset$ if $j \neq \ell$. Let $Q_{k,j}'$ be a cube contained in $Q_{k,j}$ with sides equal to δ_k', and let $V_k = \bigcup_j \mathring{Q}_{k,j}'$. We will choose δ_k so small that $m(V_k) < \varepsilon/2^k$. Let $K = \bigcap_{k=1}^\infty (E \setminus V_k)$. Then

$$m(E \setminus K) \leq \sum_{k=1}^{\infty} m(V_k) \leq \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon .$$

Now suppose $x \in K$. Then for each k there exists some j so that $x \in Q_{k,j}$. Since $Q(x, 2\delta_k) \setminus K \supset Q_{k,j}'$, we find

$$\frac{C_\alpha(Q(x, 2\delta_k) \setminus K)}{(2\delta_k)^n} > \frac{(\delta_k')^\alpha}{2^n \delta_k^n} = \frac{1}{2^n},$$

which implies

$$\limsup_{\delta \to 0} \frac{C_\alpha(Q(x, \delta) \setminus K)}{\delta^n} > 0 \qquad \text{for all } x \in K .$$

By the lemma above, $C_\alpha(B \setminus K) = C_\alpha(B)$. Hence the theorem is proved.

Now let $\{\alpha_j\}$ be increasing to n. Consider a sequence $\{K_j\}$ of compact sets contained in E satisfying $m(E \setminus K_j) < \varepsilon/2^j$ and $C_{\alpha_j}(B \setminus K_j) = C_{\alpha_j}(B)$, $j = 1, 2, \cdots$. Set $K = \bigcap_{j=1}^\infty K_j$. Then it is obvious that $m(E \setminus K) \leq \sum_{j=1}^\infty m(E \setminus K_j) < \varepsilon$, and $C_{\alpha_j}(B \setminus K) = C_{\alpha_j}(B)$ for all j. Using (ii) in the above lemma, we can verify that this implies $C_\alpha(B \setminus K) = C_\alpha(B)$ for all $\alpha \in (n - 2, n)$. Thus Theorem 4 has the following extension.

THEOREM 5. *For any compact set* E *with* $m(E) > 0$ *and for any* $\varepsilon > 0$, *there exists a compact set* $K \subset E$ *such that* $m(E \setminus K) < \varepsilon$ *and* $C_\alpha(B \setminus K) = C_\alpha(B)$ *for all* $\alpha \in (n - 2, n)$.

REFERENCES

1. L. V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null-sets*. Acta Math. 83 (1950), 101-129.

2. T. Bagby, L_p *approximation by analytic functions*. J. Approximation Theory 5 (1972), 401-404.

3. L. Carleson, *Selected Problems on Exceptional Sets*. Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J., 1967.

4. ———, *On null-sets for continuous analytic functions*. Ark. Mat. 1 (1951), 311-318.

5. T. W. Gamelin, *Uniform Algebras*. Prentice-Hall, Englewood Cliffs, N.J., 1969.

6. R. Harvey and J. Polking, *Removable singularities of solutions of linear partial differential equations*. Acta Math. 125 (1970), 39-56.

7. L. I. Hedberg, *Non-linear potentials and approximation in the mean by analytic functions*. Math. Z. 129 (1972), 299-319.

8. N. S. Landkof, *Foundations of Modern Potential Theory*. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.

9. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

10. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971.

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