THE RANGES OF ANALYTIC FUNCTIONS WITH CONTINUOUS BOUNDARY VALUES

J. Globevnik

Dedicated to my teacher Professor Ivan Vidav on the occasion of his 60th birthday

Denote by Δ, $\overline{\Delta}$, $\partial\Delta$ the open unit disc in $\mathbb{C}$, its closure, and its boundary, respectively. If X is a complex Banach space, we denote by $A(\Delta, X)$ the class of all continuous functions from $\overline{\Delta}$ to X, analytic on Δ, and we write A for $A(\Delta, \mathbb{C})$. We denote the closure and the interior of a set $S \subset X$ by $\overline{S}$ and Int S, respectively. We write $I = \{t: 0 \leq t \leq 1\}$ and denote the set of all positive integers by $\mathbb{N}$.

The main purpose of this note is to present a simple topological description of the sets $f(\overline{\Delta})$, $f \in A$. Note that the topological description of the sets $f(\partial\Delta)$, $f \in A$ is known [2].

We obtain our description by combining some ideas of Pełczyński [9] with some ideas from [6].

Definition. Let P be a subset of a metric space and let $\varepsilon > 0$. We call a finite set $S_\varepsilon \subset P$ an ε-*path-net* for P if given any $x \in P$ there exist $y \in S_\varepsilon$ and a path in P joining x and y whose diameter is less than ε. We say that P is *totally path-connected* if

(i) P is path-connected;

(ii) for every $\varepsilon > 0$ there exists an ε-path-net for P.

Remark. If P is an open subset of a Banach space, then (ii) above is equivalent to the assumption that P has "property S" [7, 12]. Note that there are bounded domains in $\mathbb{C}$ which are not totally path-connected.

THEOREM 1. *Let a subset* K *of* $\mathbb{C}$ *consist of more than one point. Then* $K = f(\overline{\Delta})$ *for some* $f \in A$ *if and only if*

(i) $K = \overline{\text{Int } K}$;

(ii) Int K *is totally path-connected.*

LEMMA 1. *Let* $F \subset \partial\Delta$ *be a closed set of Lebesgue measure* 0, *and let* $\lambda \in \partial\Delta - F$. *Assume that* $p: I \to \mathbb{C}$ *is a path satisfying* $p(0) = 0$. *Let* $\varepsilon > 0$, *and let* $U \subset \overline{\Delta}$ *be a neighborhood of* λ. *There exists* $f \in A$ *satisfying*

(i) $f(F) = \{0\}$;

(ii) $f(\lambda) = p(1)$;

(iii) $|f(x)| < \varepsilon$, $z \in \overline{\Delta} - U$;

(iv) $f(\overline{\Delta}) \subset p(I) + \varepsilon\Delta$.

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Lemma 1 is a special case of Lemma 3 in [6]. For the sake of completeness, we sketch its proof.

Proof of Lemma 1 [5, 6]. Clearly, we may assume that $F \subset \overline{\Delta} - U$. Using the Mergelyan theorem, it is easy to see that there is a polynomial P satisfying $|p(z) - P(z)| < \varepsilon/2$, $z \in I$, and $P(0) = 0$. Let $S \subset \mathbb{C}$ be an open neighborhood of I such that $P(S) \subset p(I) + (\varepsilon/2)\Delta$, and let $V \subset S$ be an open neighborhood of $I - \{1\}$ containing the point 1 in its boundary and bounded by a Jordan curve contained in S. By the Riemann mapping theorem [10], there is $\phi \in A$ mapping $\overline{\Delta}$ onto $\overline{V}$ and satisfying $\phi(0) = 0$, $\phi(1) = 1$. Let $T \subset \Delta$ be a neighborhood of 0 such that $\phi(T) \subset W$, where $W \subset V$ is a neighborhood of 0 such that $|P(z)| < \varepsilon/2$, $z \in W$. Take a sufficiently high power η of a function $\psi \in A$ satisfying $\psi(\lambda) = 1$, $|\psi(z)| < 1$, $z \in \overline{\Delta} - \{\lambda\}$ [8] such that $\eta(\overline{\Delta} - U) \subset T$, define $g = P \circ \phi \circ \eta$, and put $h(s) = -g(s)$, $s \in F$, $h(\lambda) = p(1) - g(\lambda)$. Let $\tilde{h} \in A$ be an extension of h given by the Rudin-Carleson theorem [8] satisfying $|\tilde{h}(z)| < \varepsilon/2$, $z \in \overline{\Delta}$, and put $f = g + \tilde{h}$.

Remark. Since the Mergelyan theorem and the Rudin-Carleson theorem have been generalized to the case where the range is a complex Banach space [1; 11, 4], Lemma 1 can be generalized so that p is a path in a complex Banach space X and $f \in A(\Delta, X)$ [6].

LEMMA 2. *Let* P *be a nonempty open subset of* $\mathbb{C}$ *which is totally path-connected. Let* $F \subset \partial\Delta$ *be a nonempty perfect compact set. There exists a function* $f \in A$ *such that* $f(F) = \overline{P}$ *and such that* $f(\overline{\Delta} - F) \subset P$.

Pełczyński [9] proved a similar assertion for $P = \Delta$ and for A replaced by any function algebra on a compact metric space whose set S of peak points contains a proper compact perfect subset. Below we modify carefully his idea about refining the usual net, using the fact that S contains a perfect compact set, and we use our Lemma 1 to prove Lemma 2.

Proof of Lemma 2. *Part 1.* With no loss of generality, assume that $0 \in P$. Write $F = \bigcap_{n=1}^{\infty} \mathcal{O}_n$, where $\{\mathcal{O}_n\}$ is a decreasing sequence of (relatively) open subsets of $\overline{\Delta}$. Assume that there exist a decreasing sequence $\{\varepsilon_n\}$ of positive numbers and a sequence $\{f_n\} \subset A$ with the following properties

(i) $f_n(\overline{\Delta}) + \varepsilon_n \Delta \subset P$, $\quad n \in \mathbb{N}$;

(ii) $|f_{n+1}(z) - f_n(z)| < \varepsilon_n/2^n$, $\quad z \in \overline{\Delta} - \mathcal{O}_{n+1}$, $n \in \mathbb{N}$;

(iii) $|f_{n+1}(z) - f_n(z)| < 1/2^n$, $\quad z \in \overline{\Delta}$, $n \in \mathbb{N}$;

(iv) for each $n \in \mathbb{N}$ there exists a finite set $Z_n \subset F$ such that $f_n(Z_n)$ is a $1/2^{n+1}$-path-net for P.

Observe that (i) and (iv) force $\varepsilon_n \to 0$.

Define $f = \lim f_n$. By (iii) the convergence is uniform on $\overline{\Delta}$, so $f \in A$. Since $f_n(F) \subset P$, $n \in \mathbb{N}$, $f(F) \subset \overline{P}$. Since f(F) is compact, we prove that $f(F) = \overline{P}$ by proving that f(F) is dense in P. Let $\varepsilon > 0$ and $x \in P$. Choose n sufficiently large to satisfy $|f_n(z) - f(z)| < \varepsilon/2$, $z \in \overline{\Delta}$ and $1/2^{n+1} < \varepsilon/2$. By (iv) there is some $z \in Z_n$ such that $|x - f_n(z)| < 1/2^{n+1}$, which implies that $|x - f(z)| < \varepsilon$. Consequently, f(F) is dense in P.

Now, let $z \in \overline{\Delta} - F$. There is some $n \in \mathbb{N}$ such that $z \notin \mathcal{O}_{n+1}$. Write $f(z) = f_n(z) + \sum_{k=n}^{\infty} [f_{k+1}(z) - f_k(z)]$. Since the sequences $\{\mathcal{O}_n\}$ and $\{\varepsilon_n\}$ are decreasing, it follows by (ii) that

$$\left|\sum_{k=n}^{\infty} [f_{k+1}(z) - f_k(z)]\right| < \sum_{k=n}^{\infty} \varepsilon_k/2^k < \sum_{k=n}^{\infty} \varepsilon_n/2^k < \varepsilon_n,$$

and by (i) it follows that $f(z) \in P$.

It remains to prove the existence of $\{\varepsilon_n\}$ and $\{f_n\}$ satisfying (i) through (iv) above.

Part 2. First we construct f_1. By the assumption, there exists a 1/4-path-net $\{w_1, w_2, \cdots, w_N\}$ for P. By the connectedness of P, there exists for each $k = 1, 2, \cdots, N$ a path $p_k: I \to P$ satisfying $p_k(0) = 0$, $p_k(1) = w_k$. Since $p_k(I)$, $1 \leq k \leq N$, are compact sets, there is an $\varepsilon > 0$ such that $p_k(I) + \varepsilon\Delta \subset P$, $1 \leq k \leq N$.

Choose a set of distinct points $Z_1 = \{\lambda_1, \lambda_2, \cdots, \lambda_N\} \subset F$ and a set of disjoint neighborhoods $U_i \subset \overline{\Delta}$ of the points λ_i, respectively. By Lemma 1 there exist functions $g_k \in A$, $k = 1, 2, \cdots, N$, such that

$$g_k(\lambda_j) = 0, \quad j \neq k;$$

$$g_k(\lambda_k) = w_k;$$

$$|g_k(z)| < \varepsilon/2N, \quad z \in \overline{\Delta} - U_k;$$

$$g_k(\overline{\Delta}) \subset p_k(I) + (\varepsilon/2)\Delta.$$

Define $f_1 = \sum_{k=1}^{n} g_k$. Clearly, $f_1 \in A$. By the properties of g_k, $1 \leq k \leq N$, we have $f_1(\lambda_k) = w_k$, $1 \leq k \leq N$, and consequently $f_1(Z_1)$ is a 1/4-path-net for P. If $z \in \overline{\Delta} - \bigcup_{k=1}^{N} U_k$, then $|f_1(z)| < \sum |g_k(z)| < N \cdot \varepsilon/2N = \varepsilon/2$, and consequently $f_1(z) \in P$. If $z \in U_k$ for some k, then $z \notin U_j$ for $j \neq k$. It follows that

$$f_1(z) = g_k(z) + \sum_{j \neq k} g_j(z) \in p_k(I) + (\varepsilon/2)\Delta + (\varepsilon/2)\Delta \subset P.$$

Since $f_1(\overline{\Delta})$ is a compact subset of P, there is an $\varepsilon_1 > 0$ such that $f_1(\overline{\Delta}) + \varepsilon_1 \Delta \subset P$.

Part 3. Assume that for some $n \in \mathbb{N}$ we have constructed f_n and ε_n satisfying (i) and such that there exists a finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_M\} \subset F$ for which $\{f_n(\lambda_j), j = 1, 2, \cdots, M\}$ is a $1/2^{n+1}$-path-net for P.

The proof of Lemma 2 will be completed by induction once we have shown that there exist $g \in A$ and ε_{n+1}, $0 < \varepsilon_{n+1} < \varepsilon_n$, such that

(a) $(f_n + g)(\overline{\Delta}) + \varepsilon_{n+1} \Delta \subset P$;

(b) $|g(z)| < \varepsilon_n/2^n$, $\quad z \in \overline{\Delta} - \mathcal{O}_{n+1}$;

(c) $|g(z)| < 1/2^n$, $\quad z \in \overline{\Delta}$;

(d) there exists a finite set $Z_{n+1} = \{\mu_1, \mu_2, \cdots, \mu_N\} \subset F$ such that $(f_n + g)(Z_{n+1})$ is a $1/2^{n+2}$-path-net for P.

By the assumption, there exists a $1/2^{n+2}$-path-net $\{w_1, w_2, \cdots, w_N\}$ for P. Since $\{f_n(\lambda_j), 1 \leq j \leq M\}$ is a $1/2^{n+1}$-path-net for P, we can join each w_k with some $f_n(\lambda_j)$ by a path p_k in P whose diameter is less than $1/2^{n+1}$.

Renumbering w_k - s and λ_j - s and omitting those λ_j - s whose $f(\lambda_j)$ - s are not joined with any w_k (*i.e.*, lowering M if necessary), we obtain the integers $0 = N_0 < N_1 < \cdots < N_M = N$ such that for each $j = 1, 2, \cdots, M$ and for each k, $N_{j-1} < k \leq N_j$, there exists a path $p_k: I \to P$ whose diameter is less than $1/2^{n+1}$ and such that $p_k(0) = f(\lambda_j)$, $p_k(1) = w_k$. Now, the set

$$G = f_n(\overline{\Delta}) \cup p_1(I) \cup p_2(I) \cup \cdots \cup p_N(I)$$

is compact and contained in P. Consequently, there is an $\varepsilon > 0$ such that

(1) $$G + (N+3)\varepsilon\Delta \subset P;$$

(2) $$N\varepsilon < \min\{\varepsilon_n/2^n, 1/2^n\};$$

(3) $$1/2^{n+1} + (3+N)\varepsilon < 1/2^n .$$

By the continuity of f_n, one can choose disjoint convex neighborhoods $U_j \subset \overline{\Delta}$ of the points λ_j, $1 \leq j \leq M$, respectively, which are contained in $\mathcal{O}_{n+1}$ and satisfy

(4) $$|f_n(z_1) - f_n(z_2)| < \varepsilon \qquad z_1, z_2 \in U_j, \ 1 \leq j \leq M.$$

Further, since F is a perfect set, there are distinct points $\mu_1, \mu_2, \cdots, \mu_N$ in F such that $\mu_k \in U_j$, $N_{j-1} < k \leq N_j$, $j = 1, 2, \cdots, M$. Now, define the paths

$$\tilde{p}_k(t) = \begin{cases} f_n((1-2t)\mu_k + 2t\lambda_j) & 0 \leq t \leq 1/2, \ N_{j-1} < k \leq N_j, \quad j = 1, 2, \cdots, M; \\ p_k(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Since $(1-2t)\mu_k + 2t\lambda_j \in U_j$, $0 \leq t \leq 1/2$; $N_{j-1} < k \leq N_j$, $j = 1, 2, \cdots, M$, it follows by (4) that $\tilde{p}_k(t) \in f(\lambda_j) + \varepsilon\Delta$, $N_{j-1} < k \leq N_j$, $j = 1, 2, \cdots, M$, $0 \leq t \leq 1/2$, and consequently $\tilde{p}_k(I) \subset p_k(I) + \varepsilon\Delta$, $k = 1, 2, \cdots, N$.

Choose disjoint neighborhoods $V_k \subset \overline{\Delta}$ of μ_k, $k = 1, 2, \cdots, N$, respectively, such that $V_k \subset U_j$, $N_{j-1} < k \leq N_j$, $j = 1, 2, \cdots, M$. Now Lemma 1 applies to show that there exist functions $g_k \in A$, $k = 1, 2, \cdots, N$ such that

$$g_k(\mu_j) = 0, \qquad k \neq j, \ k, j = 1, 2, \cdots, N;$$

$$g_k(\mu_k) = w_k - f_n(\mu_k), \qquad k = 1, 2, \cdots, N;$$

$$g_k(\overline{\Delta}) \subset \tilde{p}_k(I) - f_n(\mu_k) + \varepsilon\Delta, \qquad k = 1, 2, \cdots, N;$$

$$|g_k(z)| < \varepsilon, \qquad z \in \overline{\Delta} - V_k, \ k = 1, 2, \cdots, N.$$

Define $g = \sum_{k=1}^N g_k$.

Part 4. We show that g has all the required properties. First, if $z \in \overline{\Delta} - \bigcup_{k=1}^N V_k$ then

(5) $$|g(z)| \leq \sum |g_k(z)| < N\varepsilon .$$

Since $U_k \subset \mathcal{O}_{n+1}$, $1 \leq k \leq M$, it follows by (1) that (b) is satisfied. Further, by (1) and (5), $f_n(z) + g(z) \in f_n(\overline{\Delta}) + N\varepsilon\Delta \subset P$. If $z \in V_k$ for some k, then $z \notin V_i$ for

$k \neq i$. If $N_{j-1} < k \leq N_j$, we have by (4) and by (1)

$$\begin{aligned} f_n(z) + g(z) &= f_n(z) + g_k(z) + \sum_{i \neq k} g_i(z) \in f_n(z) + (\tilde{p}_j(I) - f_n(\mu_j) + \varepsilon\Delta) + N\varepsilon\Delta \\ &\subset (f_n(z) - f_n(\mu_j)) + (p_j(I) + \varepsilon\Delta) + (N+1)\varepsilon\Delta \\ &\subset \varepsilon\Delta + p_j(I) + (N+2)\varepsilon\Delta \subset P. \end{aligned}$$

Consequently, $(f_n + g)(\overline{\Delta}) \subset P$ and by the compactness of $(f_n + g)(\overline{\Delta})$, there is an ε_{n+1}, $0 < \varepsilon_{n+1} < \varepsilon_n$, such that (a) is satisfied.

Further, by the construction of the functions g_k, we have

$$g(\mu_k) = w_k - f_n(\mu_k), \qquad 1 \leq k \leq N,$$

so that $(f_n + g)(\mu_k) = w_k$, $1 \leq k \leq N$. This means that

$$\{(f_n + g)(\mu_k),\ k = 1, 2, \cdots, N\}$$

is a $1/2^{n+2}$-path-net for P, which proves (d).

Finally, if $z \in V_j$ for some j, then $z \notin V_k$ for $k \neq j$, so $|\sum_{k \neq j} g_k(z)| < N\varepsilon$. Further, we have $g_j(z) \in \tilde{p}_j(I) - f_n(\mu_j) + \varepsilon\Delta$, and since $\tilde{p}_j(0) = f_n(\mu_j)$, it follows that

$$|g_j(z)| < \operatorname{diam} \tilde{p}_j(I) + \varepsilon \leq (\operatorname{diam} p_j(I) + 2\varepsilon) + \varepsilon < 1/2^{n+1} + 3\varepsilon.$$

Consequently, $|g(z)| < (N+3)\varepsilon + 1/2^{n+1}$ which, together with (5), by (2) and by (3) implies (c).

Remark. Observe that one can prove Lemma 2 with $\mathbb{C}$ replaced by any finite-dimensional complex normed space.

Proof of Theorem 1. The "if" part follows immediately from Lemma 2. To prove the "only if" part, assume that $K = f(\overline{\Delta})$ for some $f \in A$. It is easy to see that the total path-connectedness is invariant under uniformly continuous maps, and since Δ is totally path-connected, the same is true for $f(\Delta)$. By the assumption, K consists of more than one point, which means that f is not a constant. Being analytic, f is an open map and satisfies $f(\Delta) \subset \operatorname{Int} K$. Clearly, $f(\Delta)$ is dense in K, which means that $\overline{\operatorname{Int} K} = K$. Theorem 1 will be proved once we prove the following.

LEMMA 3. *Let* P *be an open subset of a finite-dimensional normed space, and let* S *be a subset of* P *which is totally path-connected and dense in* P. *Then* P *is totally path-connected.*

Proof. It is easy to see that any ε-path-net for S is a 2ε-path-net for P.

If $K \subset \mathbb{C}$ satisfies (i) and (ii) of Theorem 1 then K consists of more than one point. By Theorem 1, $K = f(\overline{\Delta})$ for some $f \in A$. Clearly, f is not a constant, and consequently $f(\Delta) \subset \operatorname{Int} K$. Conversely, if $f\colon \overline{\Delta} \to \mathbb{C}$ is a continuous map satisfying $f(\Delta) \subset \operatorname{Int} f(\overline{\Delta})$, then the proof of Theorem 1 shows that $K = f(\overline{\Delta})$ satisfies (i) and (ii) of Theorem 1. Consequently, we have:

COROLLARY. *The class of ranges of all nonconstant functions from* A *coincides with the class of ranges of all continuous maps* $f\colon \overline{\Delta} \to \mathbb{C}$ *satisfying* $f(\Delta) \subset \operatorname{Int} f(\overline{\Delta})$.

The function f given by Lemma 2 has the property that $f(\partial\Delta) = f(\overline{\Delta}) = \overline{P}$. Lemma 2 does not give any information about $f(\Delta)$, and we ask the following.

Question. Is every open, totally path-connected subset of $\mathbb{C}$ necessarily of the form $f(\Delta)$ for some $f \in A$?

Above we have obtained a topological description of the class of ranges of functions from A. It is interesting to observe that this class is minimal in the sense that it is contained in the class of ranges of functions from any function algebra (= separating, sup-norm-closed subalgebra of C(K), containing constants) B on any uncountable compact metric space K. Indeed, Pełczyński has shown that the set of peak points for such B contains a proper compact perfect set [9, p. 657], and consequently by [9, Prop. 1] there exists $f \in B$ such that $f(K) = \overline{\Delta}$. Now if $\phi \in A$, then $\phi \circ f \in B$, since the polynomials are dense in A. Clearly, the ranges of $\phi \circ f$ and ϕ coincide.

In [5] it was proved that for any nonempty open connected set P in a separable complex Banach space X, there exists a continuous function $f: \overline{\Delta} - \{1\} \to X$, analytic on Δ and such that $f(\Delta)$ is densely contained in P. If X is infinite-dimensional, then one can never extend such an f continuously to all $\overline{\Delta}$, since Int $f(\overline{\Delta})$ is empty by the compactness of $f(\overline{\Delta})$. However, if X is finite-dimensional, the question of which open subsets of X can one fill with $f(\Delta)$, $f \in A(\Delta, X)$ densely, makes sense. Here is the answer:

THEOREM 2. *Let* P *be a nonempty open subset of a finite-dimensional complex normed space. Then there exists an* $f \in A(\Delta, X)$ *such that* $f(\Delta)$ *is densely contained in* P, *if and only if* P *is totally path-connected.*

Proof. The "if" part follows from Lemma 2 for vector-valued functions. The "only if" part follows from Lemma 3 by the fact that $f(\Delta)$, $f \in A(\Delta, X)$, is totally path-connected.

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Institute of Mathematics, Physics and Mechanics
University of Ljubljana
Ljubljana, Yugoslavia