QUASICONFORMALLY HOMOGENEOUS CURVES

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A Jordan curve C on the Riemann sphere is called *quasiconformally homogeneous* if for each pair of points P and Q $\in$ C there is a quasiconformal map ϕ defined in a neighborhood of C such that $\phi C = C$ and $\phi(P) = Q$. Examples of quasiconformally homogeneous curves are provided by the so-called quasicircles; *i.e.*, quasiconformal images of the unit circle. Other examples are not known, but the question of their existence was raised in [2] by D. K. Blevins and B. P. Palka. It is our purpose to answer this question negatively by proving the following result.

THEOREM 1. *Every quasiconformally homogeneous curve is a quasicircle.*

The proof of Theorem 1 will depend on a local characterization of quasicircles. A *Jordan domain with reference points* is a triple (D, p, p^*), where D is a Jordan domain, $p \in D$, and p^* is a point of the complementary Jordan domain D^*. A *morphism* $(D, p, p^*) \to (D_1, p_1, p_1^*)$ is a quasiconformal map f of the sphere onto itself such that $fD \subset D_1$, $f(p) = p_1$, and $f(p^*) = p_1^*$.

The *dilatation* of (D, p, p^*) is a nonnegative function Δ defined on the boundary $C = \partial D$ as follows. Let U be the open unit disc, and for $P \in C$ denote by $\mathscr{F}(P)$ the family of morphisms $f\colon (U, 0, \infty) \to (D, p, p^*)$ such that $f(1) = P$. Let K(f) denote the maximal dilatation of f, and define $\Delta(P) = \inf\{K(f)\colon f \in \mathscr{F}(P)\}$. (If $\mathscr{F}(P)$ is empty, then by convention $\Delta(P) = +\infty$.)

LEMMA. *The dilatation is a lower-semicontinuous function which assumes at least one finite value.*

Proof. For $P \in C$ let $m(P) = \liminf_{Q \to P} \Delta(Q)$; we have to show that $\Delta(P) \leq m(P)$. Suppose $m(P) < \infty$, and choose $\varepsilon > 0$. There is a sequence $\{P_i\}$ on C such that $P_i \to P$ and $\Delta(P_i) < m(P) + \varepsilon$ for each i. Choose $f_i \in \mathscr{F}(P_i)$ so that

$$K(f_i) < m(P) + \varepsilon;$$

since $\{f_i\}$ is a normal family [4, Theorem II.5.1], a subsequence converges uniformly to a morphism $f \in \mathscr{F}(P)$. Moreover, $K(f) \leq m(P) + \varepsilon$, and we conclude that $\Delta(P) \leq m(P)$.

To prove the second assertion we may assume that $p = 0$ and $p^* = \infty$, because the dilatation is invariant under Möbius transformations. Choose $P \in C$ so that the absolute value of P is as small as possible. Then the Möbius transformation $z \mapsto Pz$ is in $\mathscr{F}(P)$, hence $\Delta(P) = 1$.

We say that (D, p, p^*) is of *bounded dilatation* if Δ is a bounded function on C. If C is a quasicircle, then (D, p, p^*) and (D^*, p^*, p) are of bounded dilatation, but the converse is less obvious.

THEOREM 2. *Suppose that* (D, p, p^*) *and* (D^*, p^*, p) *are of bounded dilatation. Then the common boundary of* D *and* D^* *is a quasicircle.*

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Proof. Let Γ be an open Jordan arc in the affine plane. We say that (P_1, P_2, P_3) is a *triple on* Γ if P_1, P_2, P_3 are distinct points lying on Γ in this order. We say that Γ is of *bounded turning* if there exists a constant A such that $\overline{P_1 P_2}/\overline{P_1 P_3} \leq A$ for each triple (P_1, P_2, P_3) on Γ. Note that a Jordan curve is a quasicircle if and only if it is the union of a family of open arcs of bounded turning [4, Theorem II.8.7].

We may assume that $p = 0$ and $p^* = \infty$; then $C = \partial D$ lies in the affine plane and has a positive euclidean distance R to the origin. For $Q \in C$ let V_Q denote the open disc with center at Q and radius R/2. Since C is locally connected, there exist open Jordan arcs Γ_Q and Γ'_Q such that $Q \in \Gamma_Q \subset \Gamma'_Q \subset C \cap V_Q$ and Γ'_Q contains all points of C which are in the convex hull of Γ_Q. It remains to show that Γ_Q is of bounded turning.

Let (P_1, P_2, P_3) be a triple on Γ_Q; we may assume that $\overline{P_1 P_2} > \overline{P_1 P_3}$, since otherwise there is nothing to prove. As in [1, p. 295] we can find points P'_1 and P'_3 on the segment $P_1 P_3$ so that (P'_1, P_2, P'_3) is a triple on Γ'_Q and the segment $P'_1 P'_3$ has only its endpoints on C. Then $P'_1 P'_3$ and the subarc $P'_1 P_2 P'_3$ of Γ'_Q form the boundary of a Jordan domain $E \subset V_Q$. For definiteness, we suppose that $E \subset D$.

By hypothesis, there exists a constant K such that $\Delta(P) < K$ for each $P \in C$. In particular, there exists a K-quasiconformal morphism $f \in \mathscr{F}(P_2)$. The image of the unit circle under f is a quasicircle L which separates 0 and ∞. Thus there is a point $P_4 \in L$ such that $R < \overline{P_4 Q} < 3R/2$.

Let Γ be the longest subarc of L such that $P_4 \in \Gamma$ and Γ does not meet the closure of E. Since $fU \subset D$, the endpoints P''_1 and P''_3 of Γ lie on the segment $P'_1 P'_3$, and a simple geometric argument shows that $\overline{P_1 P_2}/\overline{P_1 P_3} \leq \overline{P''_1 P_2}/\overline{P''_1 P''_3}$. On the other hand, by a theorem of Ahlfors [1, Theorem 1], there is a constant A depending only on K such that $\overline{P''_1 P_2}/\overline{P''_1 P''_3} \leq A(\overline{P_4 P_2}/\overline{P_4 P''_3})$. Here $\overline{P_4 P_2} < 2R$ and $\overline{P_4 P''_3} > R/2$, and it follows that $\overline{P_1 P_2}/\overline{P_1 P_3} \leq 4A$. Hence Γ_Q is of bounded turning, and the proof of Theorem 2 is complete.

We proceed with the proof of Theorem 1. Let (D, p, p^*) be a Jordan domain with reference points such that the boundary of D is the given quasiconformally homogeneous curve C. In view of Theorem 2, it suffices to show that the dilatations Δ and Δ^* of (D, p, p^*) and (D^*, p^*, p) are bounded functions on C.

Let Φ^* be the family of isomorphisms $(D, p, p^*) \to (D^*, p^*, p)$, and let Φ be the family of automorphisms of (D, p, p^*). Combining the homogeneity condition with standard extension techniques, we see that for each pair of points P and $Q \in C$ there exists $\phi \in \Phi \cup \Phi^*$ such that $\phi(P) = Q$. Moreover,

$$\Delta(Q) \leq K(\phi) \cdot \Delta(P) \quad \text{or} \quad \Delta^*(Q) \leq K(\phi) \cdot \Delta(P), \tag{1}$$

according as $\phi \in \Phi$ or $\phi \in \Phi^*$.

Let F be the set of points of C at which Δ is finite, and let F^* be the corresponding set for Δ^*. Note that F and F^* are nonempty by the lemma. If $P \in F$ and $Q \in C$, then $Q \in F \cup F^*$ by (1), and we conclude that $C = F \cup F^*$. Since C is of second category in itself, it follows that at least one of the sets F and F^*, say F, is a second category subset of C.

Since Δ is semicontinuous, the set of points at which Δ is not continuous is of first category [3, Theorem 1.2]. Hence Δ is continuous at some points of F, and in particular F contains a nonempty open subset N of C. By (1) we have

$$\bigcup_{\phi \in \Phi} \phi N \subset F \quad \text{and} \quad \bigcup_{\phi \in \Phi^*} \phi N \subset F^*,$$

and by homogeneity the sets ϕN form an open covering of C. Since C is connected, we conclude that $F \cap F^*$ is nonempty. Moreover, $\phi(F \cap F^*) \subset F \cap F^*$ for each $\phi \in \Phi \cup \Phi^*$, and it follows that $F = F^* = C$.

Starting from a common point of continuity of Δ and Δ^*, we can now find a nonempty open set N such that Δ and Δ^* are bounded in N. Then Δ and Δ^* are bounded in ϕN for each $\phi \in \Phi \cup \Phi^*$, and by compactness a finite number of the sets ϕN cover C. Hence Δ and Δ^* are bounded functions on C.

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