A CLASS OF NONSPLITTABLE LINKS

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1. INTRODUCTION

In [2] and [4] it is shown that the link $L_0 = L_{01} \cup L_{02} \subset S^3 = \text{Bd}\, I^4$ (illustrated in Figure 1) does not bound disjoint smooth disks in the 4-cell I^4. To prove this, it is shown that the Arf invariant ϕ is not linear on $L_{01} \cup L_{02}$; that is,

$$\phi(L_{01} \cup L_{02}) \neq \phi(L_{01}) + \phi(L_{02}) \quad \text{Mod } 2.$$

In this paper we study the question of whether or not ϕ is linear on a given link. We are then able to determine, in Corollary 1 of Theorem 1, a class of nonsplittable links (links which do not bound disjoint planar surfaces in I^4) by showing that ϕ is not linear on each member of the class (the link $L_{01} \cup L_{02}$ is the prototype of our class).

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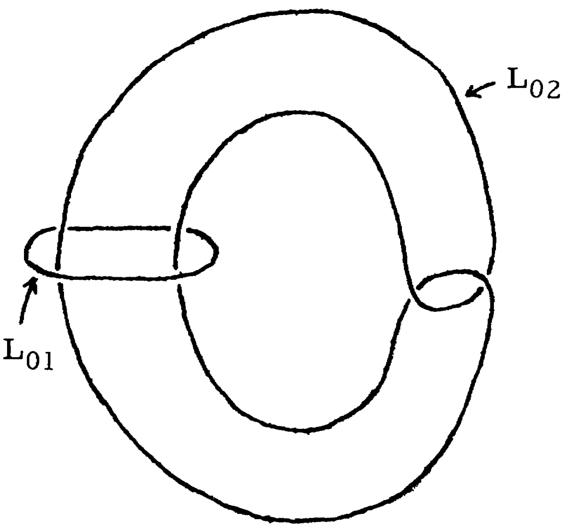

Figure 1.

2. THE *-OPERATION AND BRIDGE EQUIVALENCE

In this paper we assume all spaces and maps are piecewise linear. We call $X = \bigcup_{i=1}^{n} X_i$ a *link* if each $X_i = \bigcup_{j=1}^{n(i)} x_{ij}$, where each x_{ij} is an oriented simple closed curve in S^3, $x_{ij} \cap x_{ij'} = \emptyset$, $j \neq j'$, and $X_i \cap X_j = \emptyset$, $i \neq j$. We call the

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link X *proper* if each x_{ij} has linking number 0 Mod 2 [1, p. 81] with both of the sublinks $\bigcup_{j' \neq j} x_{ij'}$ and $\left(\bigcup_{i=1}^{n} X_i\right) - x_{ij}$. The link X is said to be *splittable* if there exist disjoint (PL) planar surfaces $D_1, \cdots, D_n$ in I^4 such that each $D_i \cap \text{Bd } I^4 = \text{Bd } D_i = X_i$. We use the Arf invariant ϕ of a knot K as defined in [6, pp. 543-544] and extend the definition of ϕ to a proper link X by defining $\phi(X) = \phi(K)$ if X is related to K [6, pp. 546-547].

We now describe the *-operation and bridge equivalence. Let $X = \bigcup_{i=1}^{n} X_i$ be a proper link and Δ a disk in S^3 such that $\Delta \cap X = \text{Bd } \Delta \cap X_i = A_{i1} \cup A_{i2}$, where A_{i1}, A_{i2} are disjoint arcs in X_i and an orientation on Δ induces an orientation on A_{i1} and A_{i2} which agrees with the orientation on A_{i1} and A_{i2} induced by the orientation on X. We now let $X_i(1) = (X_i - \Delta) \cup (\text{Bd } \Delta - \text{Int}(A_{i1} \cup A_{i2}))$, its orientation induced by the orientation on X_i, and

$$X(1) = X_1 \cup \cdots \cup X_{i-1} \cup X_i(1) \cup X_{i+1} \cup \cdots \cup X_n,$$

where X(1) was obtained from X by the *-operation if X(1) is again a proper link.

Definition 1. X is *bridge equivalent* to X', denoted $X \underset{b}{\sim} X'$, if there exists a finite sequence of proper links $X = X(0), X(1), \cdots, X(t) = X'$ such that each X(i) was obtained from X(i - 1) by the *-operation.

3. A CLASS OF NONSPLITTABLE LINKS

In this section we prove Theorem 1 and show that this theorem gives us a new and rather general class of nonsplittable links (Corollary 1 to Theorem 1).

LEMMA 1. *If* $X = \bigcup_{i=1}^{n} X_i$ *is proper and* $X \underset{b}{\sim} X' = \bigcup_{i=1}^{n} X_i'$, *then* $\phi(X) = \phi(X')$ Mod 2 *and each* $\phi(X_i) = \phi(X_i')$ Mod 2.

Proof. Since X = X(0) is related to X(1) and X(1) (or X(0) if the bridge runs between the same component of X_i) is related to some knot K, it follows from [6] that $\phi(X(0)) = \phi(X(1))$ Mod 2. By repeating this argument t times, it follows that $\phi(X) = \phi(X')$ Mod 2 and, similarly, each $\phi(X_i) = \phi(X_i')$ Mod 2.

LEMMA 2. *If* $X = \bigcup_{i=1}^{n} (X_i)$ *is a proper link and splittable, then*

$$\phi(X) = \sum_{i=1}^{n} \phi(X_i) \quad \text{Mod } 2.$$

Proof. Suppose $D_1, \cdots, D_n$ are disjoint planar surfaces in I^4 such that each $\text{Bd } D_i = X_i$. Let d_i be an arc in Int D_i which contains all the nonlocally flat points of D_i. Let a_0 be a contractible 1-complex in Int I^4 such that each

$$a_0 \cap D_i = a_0 \cap d_i$$

is exactly one endpoint of a_0. Let $b_0 = a_0 \cup \left(\bigcup_{i=1}^{n} d_i\right)$ and let $S^3 \times I$ be I^4 minus a small open regular neighborhood of b_0 where $X \subset S^3 \times 0$ and

$$\left(\bigcup_{i=1}^{n} (D_i) \right) \cap (S^3 \times 1)$$

is a link such that each knot $D_i \cap (S^3 \times 1)$ can be separated from $D_j \cap (S^3 \times 1)$ by a 2-sphere in $S^3 \times 1$, $i \neq j$. It follows from [4] and [6] that

$$\phi\left(\bigcup_{i=1}^{n} (X_i) \right) = \phi\left(\bigcup_{i=1}^{n} (D_i \cap (S^3 \times 1)) \right) = \sum_{i=1}^{n} \phi(D_i \cap (S^3 \times 1)) = \sum_{i=1}^{n} \phi(X_i) \text{ Mod } 2 .$$

Let $S = S_1 \cup S_2$ be the proper link illustrated in Figure 2. Here $S_1 = \bigcup_{j=1}^{s(1)} s_{1j}$ is $s(1)$ meridian curves of the solid torus U and $S_2 = \bigcup_{j=1}^{s(2)} s_{2j}$ is a chain of curves going around U once. The chain S_2 may twist about itself (some twisting is illustrated in the link s_{22}).

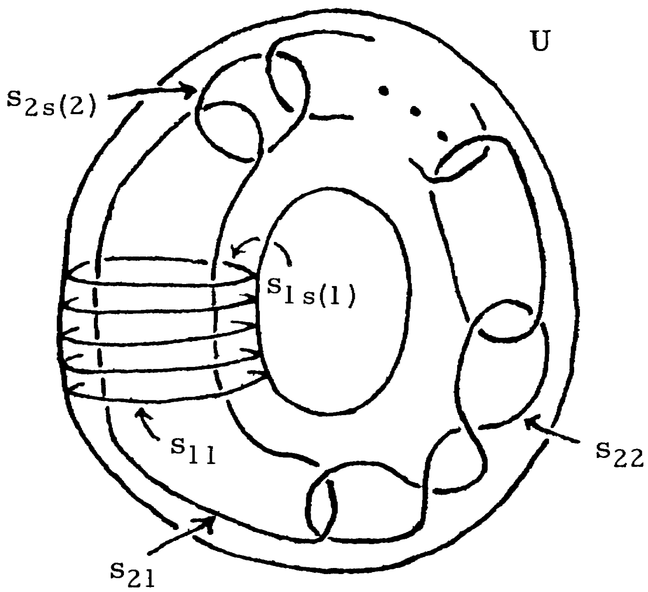

Figure 2.

LEMMA 3. *If* $s(1) = 0$ Mod 2, *then* $\phi(S_1 \cup S_2) = \phi(S_2)$ Mod 2. *If* $s(1) = 1$ Mod 2, *then* $\phi(S_1 \cup S_2) = \phi(S_2) + 1$ Mod 2.

Proof. By adding $s(2) - 1$ short bridges between adjacent links of S_2 (*-operation), it follows that $S \underset{b}{\sim} S'$, where $S_1' = S_1$ and S_2' is one curve (self-linked in U). Hence by Lemma 1, $\phi(S_1 \cup S_2) = \phi(S_1' \cup S_2')$ Mod 2 and $\phi(S_2) = \phi(S_2')$ Mod 2. It follows from an observation by Kauffman [4] that if we add a bridge from s_{1j} to S_2', then $\phi(S_2')$ changes value by one Mod 2 (*i.e.*, if we put $n = 1$, $X_1 = s_{1j} \cup S_2'$ in Section 2, then the result of the *-operation on $X = X_1$ is to put another twist in S_2'). Hence $\phi(S_1 \cup S_2) = \phi(S_1' \cup S_2') = \phi(S_2') + s(1) = \phi(S_2) + s(1)$ Mod 2.

Let T_1 be a link of one component t_{11}. For $i = 1, \cdots, m$, let U_i be an embedding in $S^3 - t_{11}$ of the solid torus U (defined in the paragraph before Lemma 3), $U_i \cap U_j = \emptyset$, $i \neq j$. Let $S_2(i)$ be the corresponding chain of curves in U_i. Let $T = T_1 \cup T_2$, where $T_2 = \bigcup_{i=1}^{m} S_2(i) = \bigcup_{j=1}^{t(2)} t_{2j}$. Denote the core circle of each U_i by $C(U_i)$.

LEMMA 4. *If the linking number of* t_{11} *with* $\bigcup_{i=1}^m C(U_i)$, *denoted by* $\ell\left(t_{11}, \bigcup_{i=1}^m C(U_i)\right)$, *is* 0 Mod 2, *then* $\phi(T_1 \cup T_2) = \phi(T_1) + \phi(T_2)$ Mod 2. *If* $\ell\left(t_{11}, \bigcup_{i=1}^m C(U_i)\right) = 1$ Mod 2, *then* $\phi(T_1 \cup T_2) = \phi(T_1) + \phi(T_2) + 1$ Mod 2.

Proof. Assume t_{11} bounds an orientable surface F in S^3 and assume F intersects each U_i in a collection of meridional disks. We now take a disjoint collection of bridges (disks) in F, where each bridge is obtained by taking a regular neighborhood in F of an arc which starts in Bd F, runs out to a point near one of the meridional disks of $F \cap \left(\bigcup_{i=1}^m U_i\right)$, goes once around this meridional disk, and then returns by a path close to the one going out. Assume we have one bridge for each meridional disk. Let $T' = T_1' \cup T_2'$ be the link obtained by applying the *-operation on each bridge. Then T_1' consists of one curve t_{11}' (the altered t_{11}), and T_2' consists of T_2 and a number of meridians of each U_i (each such meridian curve was formed from the inner part of a bridge). Let $T_2'(i)$ be the sublink of T_2' corresponding to the chain $S_2(i)$ and the meridians of U_i, so that $T_2' = \bigcup_{i=1}^m T_2'(i)$. By Lemma 1, $\phi(T) = \phi(T')$ Mod 2. Now t_{11}' bounds an orientable surface F' (what remains of F) and $F' \cap \left(\bigcup_{i=1}^m U_i\right) = \emptyset$. Regard F' as a disk with bands (see [5], for example). Note that we may pass a band of F' through some U_i at the expense of adding two more meridian curves to U_i (that is, we apply the *-operation twice to move a band through U_i). By a similar reasoning, we may untie U_i or untangle U_i from U_j at the expense of adding an even number of meridians. It follows that $T_1' \cup T_2'$ is bridge equivalent to $T_1'' \cup T_2''$, where $t_{11}' \underset{b}{\sim} t_{11}'' = T_1''$, each $T_2'(i) \underset{b}{\sim} T_2''(i)$, $T_2'' = \bigcup_{i=1}^m T_2''(i)$, and the sublinks t_{11}'', $T_2''(1), \cdots, T_2''(m)$ can be separated from each other by disjoint 2-spheres in S^3. Hence, by the above observations,

$$\phi(T_1 \cup T_2) = \phi(t_{11}'') + \sum_{i=1}^m \phi(T_2''(i)) \quad \text{Mod } 2.$$

Let $n(i)$ be the number of meridians in Bd U_i. By Lemma 3,

$$\sum_{i=1}^m \phi(T_2''(i)) = \phi(T_2) + \sum_{i=1}^m n(i) \quad \text{Mod } 2.$$

Thus $\phi(T_1 \cup T_2) = \phi(T_1) + \phi(T_2) + \sum_{i=1}^m n(i)$ Mod 2. Hence if

$$\ell\left(t_{11}, \bigcup_{i=1}^m C(U_i)\right) = 0 \quad \text{Mod } 2,$$

then $\phi(T_1 \cup T_2) = \phi(T_1) + \phi(T_2)$ Mod 2; and if $\ell\left(t_{11}, \bigcup_{i=1}^m C(U_i)\right) = 1$ Mod 2, then $\phi(T_1 \cup T_2) = \phi(T_1) + \phi(T_2) + 1$ Mod 2.

Let $Y = Y_1 \cup Y_2$ be a proper link such that $Y_1 = y_{11}$ and each component of $Y_2 = \bigcup_{j=1}^{n(2)} y_{2j}$ is null-homotopic in $S^3 - y_{11}$. Let each C_j be a singular disk in $S^3 - y_{11}$ bounded by y_{2j}. By [3], we may suppose that each C_j is the image of a nonsingular disk C_j' and that all of the singular set of $\bigcup_{j=1}^{n(2)} C_j$ was obtained by sewing an arc starting in Bd C_j' and ending in Int C_j' to an arc starting in Bd C_k' and ending in Int C_k' in such a manner that an endpoint in the boundary is sewed to an endpoint in the interior (assume also that each such arc is disjoint from the rest). We form a graph $G(Y_2)$ as follows. For each C_{2j}, let v_{2j} be a point in Int C_{2j} minus the singular set of $\bigcup_{j=1}^{n(2)} C_{2j}$. Let w_i be the midpoint of the i^{th} arc of singularities. Now form the graph $G(Y_2)$ by connecting each v_{2j} to each w_i in C_{2j} with an open arc in C_{2j} minus the singular set of $\bigcup_{j=1}^{n(2)} C_{2j}$ in such a manner that the interiors of any two such arcs are disjoint. Since Y_2 is a proper link, it follows that the degree of each vertex v_{2j} of $G(Y_2)$ is even. Now the linking number of y_{11} with $G(Y_2)$, $\ell(y_{11}, G(Y_2))$, is well defined Mod 2; it is the number of times that $G(Y_2)$ intersects a singular disk bounded by y_{11}, assuming that $G(Y_2)$ intersects and pierces the singular disk at each point of their intersection and that no vertex of $G(Y_2)$ is contained in the singular disk.

THEOREM 1. *Let* $Y = Y_1 \cup Y_2$ *be a proper link such that* $Y_1 = y_{11}$ *and each component of* $Y_2 = \bigcup_{j=1}^{n(2)} y_{2j}$ *is null-homotopic in* $S^3 - y_{11}$. *If*

$$\ell(y_{11}, G(Y_2)) = 0 \text{ Mod } 2,$$

then $\phi(Y_1 \cup Y_2) = \phi(Y_1) + \phi(Y_2)$ Mod 2. *If* $\ell(y_{11}, G(Y_2)) = 1$ Mod 2, *then* $\phi(Y_1 \cup Y_2) = \phi(Y_1) + \phi(Y_2) + 1$ Mod 2.

Proof. Let $C_1, \cdots, C_{n(2)}$ be the singular disks in $S^3 - y_{11}$ defined above. Let ζ' be a spanning arc in C_j' (*i.e.*, $\partial\zeta' = \zeta' \cap (\partial C) \subset \partial C$) such that one component of $C_j' - \zeta'$ contains two arcs of singularities and the other component contains the rest of the arcs of singularities, and let ζ be the corresponding arc in C_j. We now do the *-operation using a small regular neighborhood of ζ in C_j as the bridge to obtain a new link $Y(1)$ from Y. Repeating this process a finite number of times, we conclude that $Y \underset{b}{\sim} Y(m)$, where $Y(m) = Y_1(m) \cup Y_2(m)$, $Y_1(m) = Y_1 = y_{11}$, and $Y_2(m) \underset{b}{\sim} Y_2$; and $Y_1(m) \cup Y_2(m)$ is a link which satisfies the hypothesis of Lemma 4. By Lemma 1 and Lemma 4, we have

$$\phi(Y_1 \cup Y_2) = \phi(Y_1(m) \cup Y_2(m)) = \phi(Y_1(m)) + \phi(Y_2(m)) = \phi(Y_1) + \phi(Y_2) \text{ Mod } 2$$

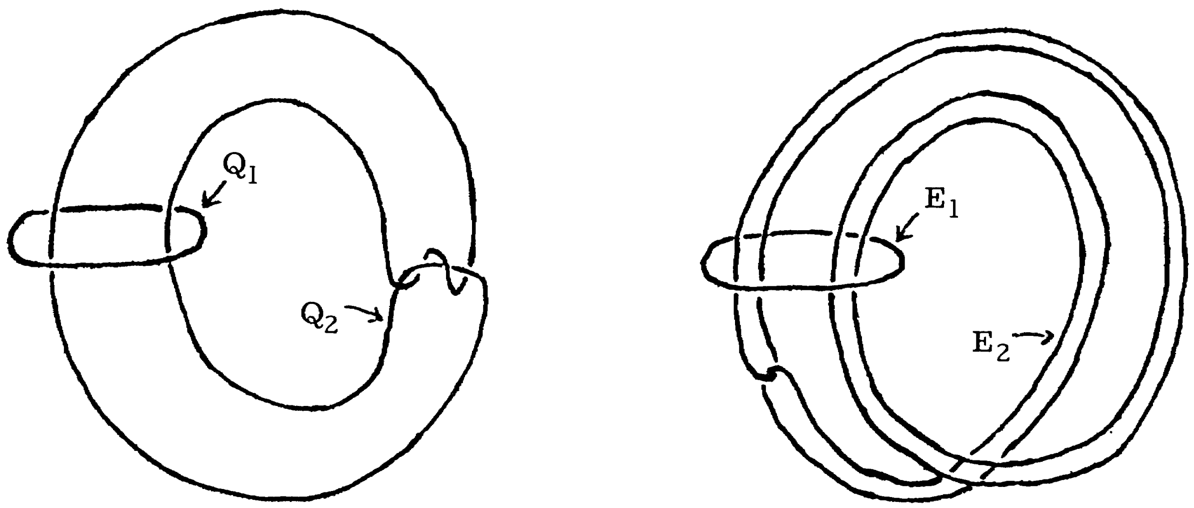

Figure 3.

if $\ell(y_{11}, G(Y_2)) = 0$ Mod 2 and, similarly, $\phi(Y_1 \cup Y_2) = \phi(Y_1) + \phi(Y_2) + 1$ Mod 2 if $\ell(y_{11}, G(Y_2)) = 1$ Mod 2.

COROLLARY 1. *If the link* $Y = Y_1 \cup Y_2$ *satisfies the hypothesis of Theorem* 1 *and* $\ell(y_{11}, G(Y_2)) = 1$ Mod 2, *then* Y *is not splittable.*

Proof. If we suppose Y is splittable, then we obtain a contradiction from Lemma 2 and Theorem 1.

Question. Is either of the links $Q_1 \cup Q_2$ or $E_1 \cup E_2$ illustrated in Figure 3 splittable? Note that $\ell(q_{11}, G(Q_2)) = \ell(e_{11}, G(E_2)) = 0$ Mod 2.

REFERENCES

1. P. S. Aleksandrov, *Combinatorial Topology.* Vol. 3, Graylock Press, Rochester, N.Y., 1956.
2. F. González-Acuña, *Dehn's construction on knots.* Bol. Soc. Mat. Mexicana 15 (1970), 58-79.
3. W. Haken, *On homotopy 3-spheres.* Illinois J. Math. 10 (1966), 159-178.
4. L. Kauffman, *An invariant of link concordance.* Topology Conference (Virginia Polytech. Inst. and State Univ., Blacksburg, Va., 1973), pp. 153-157. Lecture Notes in Math., Vol. 375, Springer, Berlin, 1974.
5. ———, *Link manifolds.* Michigan Math. J. 21 (1974), 33-44.
6. R. Robertello, *An invariant of knot cobordism.* Comm. Pure Appl. Math. 18 (1965), 543-555.

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