A CHARACTERIZATION OF NON-FIBERED KNOTS

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INTRODUCTION

A tame knot k in S^3 is *fibered* if its complement fibers over S^1. By the work of Neuwirth [5] and Stallings [7], an equivalent condition is that the commutator subgroup of $\pi_1(S^3 - k)$ be finitely generated. In this paper, we show that a certain subgroup of a knot group is its own normalizer if and only if the corresponding knot is non-fibered. To be precise, our main theorem may be stated as follows:

THEOREM. *Let* k *be a tame non-fibered knot in* S^3, *and let* F *be a minimal spanning surface* [4, Section 7] *of* k. *Let* i: $(S^3 - F) \to (S^3 - k)$ *be the inclusion map, and set* $U = i_*(\pi_1(S^3 - F)) \subseteq \pi_1(S^3 - k) = G$. *Then* U *is its own normalizer in* G.

We remark that when k is a fibered knot, the subgroup U is just G', the commutator subgroup of G, which is normal in G; in particular, since G' is proper, U is not equal to its own normalizer in this case. We also note that in any case $U \subseteq G'$. Hence, our theorem implies that when k is non-fibered, $\text{Norm}(U) = U \subsetneq G'$.

After proving our main theorem, we will use it to show that certain knots have infinitely many non-isotopic minimal spanning surfaces. More precisely, we construct, for any composite knot $K = k_1 \# k_2$, an infinite family of minimal spanning surfaces, and then, by applying our theorem to the knots k_1 and k_2, we show that if k_1 and k_2 are non-fibered, no two of the minimal spanning surfaces are ambient isotopic by an isotopy which leaves K fixed at each level.

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PROOF OF THE THEOREM

Split S^3 along F to obtain a manifold whose boundary consists of two copies of F, say F_1 and F_2. The inclusions of F_1 and F_2 into this manifold induce homomorphisms $f_1: \pi_1(F) \to \pi_1(S^3 - F)$ and $f_2: \pi_1(F) \to \pi_1(S^3 - F)$. Since F is minimal, both f_1 and f_2 are injective, by Dehn's lemma and the loop theorem [5, p. 28]. If either f_1 or f_2 were surjective, then, by the Brown product theorem [1] (see also [7, Sections 6-10]), $(S^3 - F) = (\text{int } F) \times [0, 1]$, so that k would be a fibered knot. Therefore, since k is non-fibered, neither f_1 nor f_2 is surjective.

If we set $G = \pi_1(S^3 - k)$, $H = \pi_1(S^3 - F)$, and $A = f_1(\pi_1(F))$, and if we let ϕ be the isomorphism $f_2 \circ f_1^{-1}$ between $A = f_1(\pi_1(F))$ and $\phi(A) = f_2(\pi_1(F))$, then Van Kampen's theorem implies that G is the HNN group

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$$G = \{H, t: t^{-1} at = \phi(a) \text{ for all } a \in A\}.$$

Also, our subgroup $U = i_*(\pi_1(S^3 - F)) \subseteq G$ is just the group H (regarded, naturally, as a subgroup of the HNN group G). Finally, the fact that neither f_1 nor f_2 is surjective implies that A and $\phi(A)$ are both proper subgroups of H.

Using Serre's construction in [6], we can find a tree Γ on which the HNN group G acts, such that there is a vertex $v \in \Gamma$ whose stabilizer I_v is H, and for every edge e incident to v, the stabilizer I_e of e is a conjugate in H of either A or $\phi(A)$. Since both A and $\phi(A)$ are proper subgroups of H, we have then that $I_e \subsetneq I_v$ for each edge e incident to v.

Now take $g \in G - I_v$. Then $gv \neq v$, and $gI_v g^{-1} = I_{gv}$. Thus if g normalized I_v, then I_v would stabilize both v and gv, and hence, since Γ is a tree, I_v would stabilize all edges and vertices on the unique path between v and gv. In particular, I_v would stabilize some edge e incident to v, contradicting the fact that for such an edge e, $I_e \subsetneq I_v$. Therefore, no $g \in G - I_v$ can normalize I_v, so that $U = H = I_v$ is its own normalizer in G.

AN APPLICATION

The theorem we have just proved is a useful tool for distinguishing between spanning surfaces of knots. To illustrate this, we shall reprove the result established in [3]: if $K = k_1 \# k_2$ [4, Section 7], where k_1 and k_2 are non-fibered knots, then K has an infinite collection of minimal spanning surfaces, no two of which are (ambient) isotopic by an isotopy which leaves K fixed at each level.

Indeed, let K be the composite of two non-fibered knots k_1 and k_2. Then we may take a 2-sphere S^2 dividing S^3 into two 3-balls B_1 and B_2, and an arc $a \subseteq S^2$ such that K intersects S^2 in ∂a (= two points), $(K \cap B_1) \cup a$ is the knot k_1, and $(K \cap B_2) \cup a$ is the knot k_2. Take minimal spanning surfaces F_1 and F_2 for k_1 and k_2, respectively, with $F_1 \cap B_2 = F_2 \cap B_1 = a$, and set $F = F_1 \cup F_2$, which is a minimal spanning surface for K (see [4, Section 7]). Take a point $x \in (S^2 - a)$, and let $R: S^2 \times I \to S^2$ be an isotopic deformation of S^2 which leaves ∂a fixed at each level and takes a to itself, such that $R(x \times I)$ is a closed path representing a generator $\zeta \in \pi_1(S^2 - \partial a, x) \cong \mathbb{Z}$. Extend R to an isotopic deformation E of B_1 which leaves $(K \cap B_1)$ fixed at each level. Then $(E_1 \mid B_1 - K)_*$ is the inner automorphism of $\pi_1(B_1 - K, x)$ given by $\eta \mapsto \zeta^{-1} \eta \zeta$. (We let ζ denote its own image under the maps induced by the inclusions $(S^2 - K) \to (B_1 - K)$, $(S^2 - K) \to (B_2 - K)$, and $(S^2 - K) \to (S^3 - K)$.) For each integer j, set $F_1^j = (E_1)^j(F_1)$ and $F^j = F_1^j \cup F_2$, which is again a minimal spanning surface for K.

Let $i_1^j: (B_1 - F_1^j) \to (B_1 - K)$ $(j \in \mathbb{Z})$, $i_2: (B_2 - F_2) \to (B_2 - K)$, and $i^j: (S^3 - F^j) \to (S^3 - K)$ $(j \in \mathbb{Z})$ be inclusion maps, and let

$$U_1^j = (i_1^j)_*(\pi_1(B_1 - F_1^j, x)) \subseteq \pi_1(B_1 - K, x) \quad (j \in \mathbb{Z}),$$

$$U_2 = (i_2)_*(\pi_1(B_2 - F_2, x)) \subseteq \pi_1(B_2 - K, x),$$

and

$$U^j = (i^j)_*(\pi_1(S^3 - F^j, x)) \subseteq \pi_1(S^3 - K, x) \quad (j \in \mathbb{Z}).$$

Since $F_1^j = (E_1)^j(F_1)$, and $(E_1 \mid B_1 - K)_*$ is the inner automorphism of $\pi_1(B_1 - K, x)$ given by conjugation by ζ^{-1}, we see that $U_1^j = \zeta^{-j} U_1^0 \zeta^j$; for arbitrary integers j and ℓ, we have $U_1^j = \zeta^{-j} U_1^0 \zeta^j = \zeta^{-(j-\ell)}(\zeta^{-\ell} U_1^0 \zeta^{\ell})\zeta^{(j-\ell)} = \zeta^{(\ell-j)} U_1^{\ell} \zeta^{-(\ell-j)}$.

Note that $\pi_1(B_1 - K, x)$ is naturally isomorphic to $\pi_1(S^3 - k_1, x)$ in such a way that, for each j, U_1^j corresponds to the image of $\pi_1(S^3 - F_1^j, x)$ in $\pi_1(S^3 - k_1, x)$ under the inclusion map of $(S^3 - F_1^j)$ into $(S^3 - k_1)$. Similarly, $\pi_1(B_2 - K, x)$ is naturally isomorphic to $\pi_1(S^3 - k_2, x)$ in such a way that U_2 corresponds to the image of $\pi_1(S^3 - F_2, x)$ in $\pi_1(S^3 - k_2, x)$ under the inclusion map of $(S^3 - F_2)$ into $(S^3 - k_2)$. Thus, we may apply our theorem to conclude that, for each j, $\mathrm{Norm}(U_1^j) = U_1^j \subseteq (\pi_1(B_1 - K, x))'$, and that $\mathrm{Norm}(U_2) = U_2 \subseteq (\pi_1(B_2 - K, x))'$.

Also, $\pi_1(S^3 - K, x)$ is a free product with amalgamation

$$\pi_1(B_1 - K, x) \underset{\mathbb{Z}}{*} \pi_1(B_2 - K, x),$$

and, using the argument in [2], we can find homomorphisms

$$\phi_1\colon \pi_1(S^3 - K, x) \to \pi_1(B_1 - K, x) \quad \text{and} \quad \phi_2\colon \pi_1(S^3 - K, x) \to \pi_1(B_2 - K, x)$$

such that $\phi_1 \mid (\pi_1(B_1 - K, x)) = \mathrm{id}$, ϕ_1 kills $(\pi_1(B_2 - K, x))'$, $\phi_2 \mid (\pi_1(B_2 - K, x)) = \mathrm{id}$, and ϕ_2 kills $(\pi_1(B_1 - K, x))'$. Here we are regarding $\pi_1(B_1 - K, x)$ and $\pi_1(B_2 - K, x)$ as subgroups of $\pi_1(S^3 - K, x) = \pi_1(B_1 - K, x) \underset{\mathbb{Z}}{*} \pi_1(B_2 - K, x)$. Plainly, U^j is the subgroup of $\pi_1(S^3 - K, x)$ generated by $U_1^j \subseteq \pi_1(B_1 - K, x)$ and by $U_2 \subseteq \pi_1(B_2 - K, x)$. Since $\phi_1(U_1^j) = U_1^j$ and $\phi_1(U_2) = 0$, while $\phi_2(U_1^j) = 0$ and $\phi_2(U_2) = U_2$, we have that $\phi_1(U^j) = U_1^j$, while $\phi_2(U^j) = U_2$. We note also that if $n_1\colon \pi_1(B_1 - K, x) \to \mathbb{Z}$ and $n_2\colon \pi_1(B_2 - K, x) \to \mathbb{Z}$ are the abelianization maps which take ζ to $1 \in \mathbb{Z}$, then $n_1 \circ \phi_1$ and $n_2 \circ \phi_2$ are also abelianization maps, since they both map $\pi_1(S^3 - K, x)$ onto $\mathbb{Z}$. Thus, since

$$n_1 \circ \phi_1(\zeta) = n_1(\zeta) = 1 = n_2(\zeta) = n_2 \circ \phi_2(\zeta),$$

we have that $n_1 \circ \phi_1 = n_2 \circ \phi_2$.

Now suppose that F^{ℓ} is isotopic to F^j by an isotopy which leaves K fixed at each level. This implies that U^{ℓ} is conjugate to U^j (see [2], [3]), say $U^{\ell} = \xi U^j \xi^{-1}$ where $\xi \in \pi_1(S^3 - K, x)$. Then

$$U_2 = \phi_2(U^{\ell}) = \phi_2(\xi U^j \xi^{-1}) = (\phi_2(\xi))(\phi_2(U^j))(\phi_2(\xi))^{-1} = (\phi_2(\xi))U_2(\phi_2(\xi))^{-1}.$$

Since $\mathrm{Norm}(U_2) \subseteq (\pi_1(B_2 - K, x))'$, it follows that $\phi_2(\xi) \in (\pi_1(B_2 - K, x))'$, so $n_2 \circ \phi_2(\xi) = 0$. On the other hand,

$$\begin{aligned} U_1^j &= \zeta^{(\ell-j)} U_1^{\ell} \zeta^{-(\ell-j)} = \zeta^{(\ell-j)}(\phi_1(U^{\ell}))\zeta^{-(\ell-j)} = \zeta^{(\ell-j)}(\phi_1(\xi U^j \xi^{-1}))\zeta^{-(\ell-j)} \\ &= \zeta^{(\ell-j)}(\phi_1(\xi))(\phi_1(U^j))(\phi_1(\xi))^{-1}\zeta^{-(\ell-j)} = (\zeta^{(\ell-j)}\phi_1(\xi))U_1^j(\zeta^{(\ell-j)}\phi_1(\xi))^{-1}. \end{aligned}$$

Since $\mathrm{Norm}(U_1^j) \subseteq (\pi_1(B_1 - K, x))'$, it follows that $\zeta^{(\ell-j)}\phi_1(\xi) \in (\pi_1(B_1 - K, x))'$, so

$n_1(\zeta^{(\ell-j)}\phi_1(\xi)) = 0$, and hence $n_1 \circ \phi_1(\xi) = j - \ell$. However, $n_1 \circ \phi_1(\xi) = n_2 \circ \phi_2(\xi)$, so $j - \ell = 0$, or $j = \ell$. Thus, F^ℓ is isotopic to F^j by an isotopy which leaves K fixed at each level only if $\ell = j$.

Therefore, K has an infinite collection of minimal spanning surfaces, no two of which are isotopic by an isotopy which leaves K fixed at each level, as desired.

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