# THE INTERSECTION PAIRING ON A HOMOGENEOUS KÄHLER MANIFOLD

Frank Connolly and Tadashi Nagano

Our main result is the computation of the intersection pairing on any compact homogeneous Kähler manifold, reducing the calculation to Lie algebra and Weyl group invariants. In particular, we show the signature of such a manifold is greater than or equal to 0.

It is known (see Matsushima [6]) that if M is a compact homogeneous Kähler manifold, then M is a product of a torus and a simply connected compact homogeneous Kähler manifold. This reduces us to the case where M is 1-connected, in which case M has the form G/P for some semisimple complex Lie group G and parabolic subgroup P (see [6]).

The main idea is to take the Bruhat decomposition for G/P and compute the Poincaré duality there. We show that the Poincaré dual of each "Schubert cell" is a Schubert cell. (See Corollary 2.6.) From this, the calculation is not hard. We conclude:

MAIN THEOREM. *Let* G *be a complex connected semisimple Lie group,* P *a parabolic subgroup, with Weyl groups* W *and* $W_P$*, respectively. Let* $D \in W$ *be the unique element sending each positive root to a negative root. Let* $N(W_P)$ *be the normalizer of* $W_P$ *in* W*. Then the index of* G/P *is* $|N(W_P)/W_P| \cdot c$*, where* c *is the number of subgroups of* W *which are conjugate to* $W_P$ *and contain* D*. Moreover, the matrix of the intersection pairing* $H_*(G/P; Z) \otimes H_*(G/P; Z) \to Z$ *has the form*

$$\begin{bmatrix} I_q & 0 \\ 0 & \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \end{bmatrix}, \quad \textit{where } q = \text{index}(G/P).$$

## I. NOTATION AND CONVENTIONS

We begin by summarizing notation and recalling the Bruhat decomposition. The Lie algebra background can be obtained from [3]. Let G be a complex semisimple Lie group, with Lie algebra g, maximal toral subalgebra t, Borel subalgebra $b \supset t$, root system $\Phi$ (relative to t), and root space decomposition $g = t + \sum_{\alpha \in \Phi} g_\alpha$. Let $\Phi_+$ and $\Phi_-$ denote the positive and negative roots of $\Phi$. Thus, $b = \sum_{\alpha \geq 0} g_\alpha$ (here $g_0$ means t). Let B and T be the Lie subgroups of G corresponding to b and t, and let P be any closed connected subgroup of G containing B (a parabolic subgroup). Let G/P denote the space of left cosets.

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Let W and $W_P$ denote the Weyl groups of G and P, respectively. That is, $W = N(T)/T$, $W_P = P \cap N(T)/T \subset W$. Recall that W acts on the right on $\Phi$, and if $w \in W$, and $\alpha, \beta \in \Phi$, then $\alpha^w = \beta$ if and only if $w^{-1} g_\alpha w = g_\beta$.

Let $D \in W$ be the unique element such that $\Phi_-^D = \Phi_+$. The existence and uniqueness of D follow from the fact that W acts simply and transitively on the Weyl chambers, one of which contains $\Phi_+$ while its negative contains $\Phi_-$. For the same reason, it is clear that $D^2 = 1$.

Let $W/W_P$ be the set of left cosets of $W_P$ in W. Recall that to each $\sigma \in W$ there corresponds a number $\ell(\sigma)$, equal to $|\Phi_+^\sigma \cap \Phi_-|$, called the *minimal length* of $\sigma$. Analogously, for each $x \in W/W_P$ we define $\ell(x) = \min\{\ell(\sigma): \sigma W_P = x\}$.

The Bruhat decomposition [1], as improved by Chevalley [2] and Kostant [5], can be summarized as follows.

THEOREM 1.1. *For each* $x \in W/W_P$, *set* $e_x = BxP \bmod P$ *in* G/P. *Then:*

(1) $G/P = \bigcup \{e_x: x \in W/W_P\}$ *and* $e_x \cap e_y = \emptyset$ *unless* $x = y$;

(2) *Each* $e_x$ *is an analytic submanifold of* G/P, *isomorphic to* $\mathbb{C}^\ell$, *where* $\ell = \ell(x)$;

(3) $\bar{e}_x - e_x$ *is a union of certain* $e_y$ *such that* $\ell(y) < \ell(x)$.

For (1), see [5, p. 123, Proposition 6.1]; for (2), see [5, p. 126, Proposition 6.3]; and for (3), see [5, p. 127, equations 6.4.2 and 6.4.3].

The closure here can be taken in either the Hausdorff or Zariski sense (see [5]). A proof of (1) and (2), all quite neat, appears in [4, p. 171].

Thus G/P is endowed with a "cell decomposition". We shall also be interested in a second cell decomposition. For each $x \in W/W_P$, let $f_x = B^D xP \bmod P \subset G/P$. Here $B^D = D^{-1} BD$, which is well defined since $T \subset B$. Note that $f_x$ is an analytic manifold isomorphic to $e_{Dx}$. The isomorphism is given as follows. Let $\delta \in N(T)$ be such that $\delta \cdot T = D$. Since G acts on the left on G/P, $\delta$ determines an isomorphism of varieties $\delta: G/P \to G/P$, and $\delta \cdot e_{Dx} = \delta BDxP = B^D xP = f_x$, as claimed.

We now see easily:

COROLLARY 1.2. $G/P = \bigcup \{f_x: x \in W/W_P\}$. *This union is disjoint, each* $f_x$ *is isomorphic to* $\mathbb{C}^q$, $q = \ell(Dx)$, *and* $\bar{f}_x - f_x$ *is a union of cells* $f_y$ *such that* $\ell(Dy) \leq q$.

We shall see that these two cell decompositions of G/P are dual in a very precise way.

## II. THE DUALITY THEOREM

Our conventions are as above. Let $\alpha \in \Phi$ be a root. Define

$$g_{(\alpha, -\alpha)} = g_\alpha + g_{-\alpha} + [g_\alpha, g_{-\alpha}].$$

Set $b_\alpha = g_\alpha + [g_\alpha, g_{-\alpha}]$. Then $b_\alpha$ and $g_{(\alpha, -\alpha)}$ are subalgebras of g, $g_{(\alpha, -\alpha)}$ is semisimple, and $B_\alpha$ is a Borel subgroup if $B_\alpha = \exp b_\alpha$. Write $G_\alpha$ for $\exp g_\alpha$ and $G_{(\alpha, -\alpha)}$ for $\exp g_{(\alpha, -\alpha)}$. The reflection $\sigma_\alpha$ in $W = N(T)/T$ is then the unique nonzero element in the image of $N(T) \cap G_{(\alpha,-\alpha)}/T \cap G_{(\alpha,-\alpha)} \to N(T)/T$, and we will write $s_\alpha$ for the unique element which maps to $\sigma_\alpha$.

LEMMA 2.1. $G_\alpha s_\alpha \subset (G_{-\alpha} \cup s_\alpha)B_\alpha$ *for all* $\alpha \in \Phi$.

*Proof.* The Bruhat decomposition (see Theorem 1.1) for $G_{(\alpha,-\alpha)}$ says that

$$\begin{aligned} G_{(\alpha,-\alpha)} &= B_\alpha \{1 \cup s_\alpha\} B_\alpha \qquad \text{(here } B = P = B_\alpha) \\ &= B_\alpha \cup B_\alpha s_\alpha B_\alpha \\ &= B_\alpha \cup s_\alpha B_{-\alpha} B_\alpha \qquad \text{(since } B_\alpha^{\sigma_\alpha} = B_{-\alpha}) \\ &= B_\alpha \cup s_\alpha G_{-\alpha} B_\alpha \qquad \text{(since } B_{-\alpha} = G_{-\alpha}[G_\alpha, G_{-\alpha}]). \end{aligned}$$

Thus, $G_{(\alpha,-\alpha)} = s_\alpha G_{(\alpha,-\alpha)} = s_\alpha B \cup G_{-\alpha} B_\alpha$. Hence

$$G_\alpha s_\alpha \subset G_{(\alpha,-\alpha)} = (s_\alpha \cup G_{-\alpha})B_\alpha .$$

Let $\Delta$ denote the base of $\Phi$ given by B. Then each element of $\Phi_+$ has a unique expression as a positive integral combination of elements of $\Delta$.

LEMMA 2.2. *Let* $\alpha \in \Delta$. *Then:*

(a) $B\sigma_\alpha \subset (G_{-\alpha} \cup \sigma_\alpha)B$;

(b) $\sigma_\alpha B^D \subset B^D(G_\alpha \cup \sigma_\alpha)$.

*Proof.* We prove (a) only; the proof of (b) is virtually identical. Let $b'_\alpha = +\sum \{g_\beta : \beta \in \Phi^+ - \{\alpha\}\}$. Since $\alpha$ is not a sum of elements of $\Phi^+$, the identity $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$ shows that $b'_\alpha$ is an ideal of b; and $b = g_\alpha + b'_\alpha$, so $B = G_\alpha B'_\alpha$, where $B'_\alpha = \exp b'_\alpha$. It is well known that $(b'_\alpha)^{\sigma_\alpha} = b'_\alpha$, whence $\sigma_\alpha B'_\alpha = B'_\alpha \sigma_\alpha$. Therefore, by Lemma 2.1,

$$B\sigma_\alpha = G_\alpha B'_\alpha \sigma_\alpha = G_\alpha \sigma_\alpha B'_\alpha \subset (G_{-\alpha} \cup \sigma_\alpha)B_\alpha B'_\alpha = (G_{-\alpha} \cup \sigma_\alpha)B,$$

as claimed.

*Notation.* Let $I = (\alpha_1, \alpha_2, \cdots, \alpha_r)$ be an ordered r-tuple of elements of $\Delta$. Then $\sigma_I$ means $\sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_r}$. Also, $J < I$ means that $J = (\alpha_{j_1}, \cdots, \alpha_{j_s})$, where $0 < j_1 < \cdots < j_s \leq r$ and $s < r$.

PROPOSITION 2.3. *For any* $\sigma_I$ *in* W, $B\sigma_I B \subset \sigma_I B \cup \bigcup_{J<I} B^D \sigma_J B$.

*Proof.* We use induction on r, where $I = (\alpha_1, \alpha_2, \cdots, \alpha_r)$. If $r = 0$, so that $\sigma_I = 1$, the result is trivial. Thus we assume the theorem is true if the "length" of I is less than r, and we turn to $B\sigma_I B$, where $I = (\alpha_1, \cdots, \alpha_r)$. Let

$$I' = (\alpha_2, \alpha_3, \cdots, \alpha_r).$$

Thus

$$B\sigma_I B = B\sigma_{\alpha_1}\sigma_{I'}B \subset (G_{-\alpha_1} \cup \sigma_{\alpha_1})B\sigma_{I'}B \qquad \text{(by Lemma 2.2)}$$

$$\subset (G_{-\alpha_1} \cup \sigma_{\alpha_1})\Big(\sigma_{I'}B \cup \bigcup_{J<I'} B^D\sigma_J B\Big) \qquad \text{(by induction)}$$

$$\subset B^D\sigma_{I'}B \cup \sigma_I B \cup \bigcup_{J<I'} \{B^D\sigma_J B \cup \sigma_{\alpha_1} B^D \sigma_J B\} \qquad \text{(since } G_{-\alpha_1} \subset B^D)$$

$$\subset \Big(\bigcup_{J<I} B^D\sigma_J B\Big) \cup \sigma_I B \cup \bigcup_{J<I'} B^D(G_{\alpha_1} \cup \sigma_{\alpha_1})\sigma_J B \qquad \text{(by Lemma 2.2 (b))}$$

$$\subset \Big(\bigcup_{J<I} B^D\sigma_J B\Big) \cup \sigma_I B \cup \bigcup_{J<I'} \{B^D B\sigma_J B \cup B^D\sigma_{\alpha_1}\sigma_J B\} \qquad \text{(since } G_{\alpha_1} \subset B)$$

$$\subset \Big(\bigcup_{J<I} B^D\sigma_J B\Big) \cup \sigma_I B \cup \bigcup_{J\leq I'} B^D(B^D\sigma_J B) \qquad \text{(by induction on J)}$$

$$= \sigma_I B \cup \bigcup_{J<I} B^D\sigma_J B,$$

as claimed.

COROLLARY 2.4. *For any* $x \in W/W_P$, *we have*

$$\bar{e}_x \subset xP \cup \bigcup \{f_y\colon \ell(y) < \ell(x),\ y \in W/W_P\}.$$

*Proof.* Let $\sigma \in x$ be such that $\ell(\sigma) = \ell(x)$. By [3, p. 51], $\ell(\sigma) \leq q$ if and only if $\sigma = \sigma_I$ for $I = (\alpha_1, \alpha_2, \cdots, \alpha_r)$, $r \leq q$. So write $\sigma = \sigma_I$, where $I = (\alpha_1, \cdots, \alpha_r)$, $r = \ell(\sigma)$. Thus $BxP = B\sigma_I P$. Now by the previous proposition,

$$B\sigma_I B \subset \sigma_I B \cup \bigcup \{B^D\sigma_J B\colon J < I\},$$

so in particular, $B\sigma B \subset \sigma B \cup \bigcup\{B^D\tau B\colon \ell(\tau) < \ell(\sigma)\}$. Multiply on the right by P now, and get: $B\sigma P \subset \sigma P \cup \bigcup\{B^D\tau P\colon \ell(\tau) < \ell(x)\}$.

Since $\ell(\tau) \geq \ell(\tau W_P)$ for all $\tau$, we get

$$BxP \subset xP \cup \bigcup\{B^D yP\colon y \in W/W_P \text{ and } \ell(y) < \ell(x)\}.$$

So $e_x \subset xP \cup \bigcup\{f_y\colon y \in W/W_P \text{ and } \ell(y) < \ell(x)\}$. But $\bar{e}_x - e_x \subset \bigcup\{e_z\colon \ell(z) < \ell(x)\}$, by Theorem 1.1. Since $zP \subset f_z$, the result now follows by induction on $\ell(x)$.

DUALITY THEOREM 2.5. *For each* x *in* W, *we have*

(a) $e_x \cap f_x = \bar{e}_x \cap \bar{f}_x$ = *a single point; the intersection is transverse and the intersection number is* $+1$;

(b) $\bar{e}_x \cap \bar{f}_z = \emptyset$ *if* $\ell(x) \leq \ell(z)$ *but* $x \neq z$.

*Proof.* Suppose $x, z \in W/W_P$ and $\ell(x) \leq \ell(z)$. By Corollary 2.4,

$$\bar{e}_x \subset xP \cup \bigcup \{f_y \colon \ell(y) < \ell(x)\}.$$

Therefore,

$$\bar{e}_x \cap f_z \subset (xP \cap f_z) \cup \bigcup \{f_y \cap f_z \colon \ell(y) < \ell(z)\},$$

so that $\bar{e}_x \cap f_z = xP \cap f_z$. But $xP \in B^D xP \cap BxP = e_x \cap f_x$, so we get

$$(*) \qquad \begin{cases} \bar{e}_x \cap f_z = \emptyset & \text{if } z \neq x; \\ \bar{e}_x \cap f_x = e_x \cap f_x = xP. \end{cases}$$

Next we show that $e_x$ and $f_x$ intersect transversally at xP. Notice that $g = b + b^D$. Hence, B and $B^D$ intersect transversally at the point $e \in G$. It follows that if $s \in N(T)$ is in the coset of x, then $B \cdot s$ and $B^D \cdot s$ intersect in G transversally at s. Hence, we see that, in G/P, the manifolds BsP and $B^D sP$ intersect transversally, as claimed.

Note that $e_x$ and $f_x$ are complex submanifolds of G/P, so their intersection number at xP must be +1 rather than -1. Also, they intersect at a single point transversally, so we see $\ell(x) = \dim e_x = \dim(G/P) - \dim(f_x) = \dim(G/P) - \dim(e_{Dx})$. It follows that $\ell(x) = \dim(G/P) - \ell(Dx)$. Thus, $\ell(z) < \ell(y)$ if and only if $\ell(Dz) > \ell(Dy)$. By Corollary 1.2, then, we conclude: $\bar{f}_z - f_z \subset \bigcup \{f_y \colon \ell(y) > \ell(z)\}$.

In view of (*), we see $\bar{e}_x \cap (\bar{f}_z - f_z) = \emptyset$ if $\ell(x) \leq \ell(z)$. Thus (*) can be sharpened to say

$$\bar{e}_x \cap \bar{f}_z = \begin{cases} \emptyset & \text{if } \ell(x) \leq \ell(z),\ x \neq z; \\ xP & \text{if } x = z. \end{cases}$$

This proves the theorem.

In $H_*(G/P; Z)$, let $[e_x]$ denote the homology class represented by the even-dimensional cell $e_x$. Clearly, these classes $[e_x]$, $x \in W/W_P$, are a basis for $H_*(G/P; Z)$, and so they determine a dual basis $[e^x]$ in $H^*(G/P; Z)$ (if $[e_x] \in H_{2i}$, then $[e^x] \in H^{2i}$).

COROLLARY 2.6. *For each* x *in* $W/W_P$, *the Poincaré dual of* $[e_x]$ *is* $[e^{Dx}]$.

*Proof.* Let $\cdot$ denote the intersection product in $H_*(G/P; Z)$, and let $[f_x]$ denote the homology class determined by $f_x$. Since $\bar{e}_x$ and $\bar{f}_y$ can be taken as the closure in the Zariski sense, it is clear from the duality theorem that

$$[e_x] \cdot [f_z] = \begin{cases} 1 & \text{if } z = x; \\ 0 & \text{if } z \neq x \text{ and } \ell(x) \leq \ell(z). \end{cases}$$

But $[f_z] = \delta_*[e_{Dz}]$ and $\delta_* = 1$, since $\delta \in G$ and G is connected. Therefore,

$$[e_x] \cdot [e_y] = \begin{cases} 1 & \text{if } y = Dx; \\ 0 & \text{if } y \neq Dx \text{ and } \ell(y) \leq \ell(Dx). \end{cases}$$

Thus, $[e_x] \cdot [e_y] = 0$ if $y \neq Dx$. Also, $\dim [e_y] = 2\dim_R (G/P) - \dim [e_x]$. But by the Duality Theorem (a), this last number is $\dim [e_{Dx}] = \ell[Dx]$. The result follows at once.

## III. THE MAIN THEOREM

*Proof of the Main Theorem.* The signature of G/P is the Sylvester index of the intersection form $H_*(G/P; R) \otimes H_*(G/P; R) \to R$.

Write the elements of $W/W_P$ as $x_1, \cdots, x_q, y_1, \cdots, y_r, y_1', \cdots, y_r'$, where these are distinct, and $x_i = Dx_i$, $y_i' = Dy_i$. Possibly r or q is 0. Relative to this ordering of the basis for $H_*(G/P; Z)$, the matrix of the intersection form is:

$$\begin{bmatrix} I_q & 0 \\ 0 & \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \end{bmatrix},$$

by Corollary 2.6. This proves the second part of the main theorem. Also, it is clear that

$$\begin{aligned} \sigma(G/P) = q &= |\{x \in W/W_P: Dx = x\}| = |\{x \in W/W_P: D \in xW_Px^{-1}\}| \\ &= |\{\xi \in W/N(W_P): D \in \xi W_P \xi^{-1}\}| \cdot |N(W_P)/W_P| \\ &= |\{H \subset W: H \text{ is conjugate to } W_P \text{ and } D \in H\}| \cdot |N(W_P)/W_P|, \end{aligned}$$

as claimed.

COROLLARY 3.1. *Each compact homogeneous Kähler manifold* M *has signature greater than or equal to* 0.

*Proof.* As mentioned above, either $M = G/P$ or $M = G/P \times T$, where T is a torus; in this latter case $\sigma(M)$ is clearly 0, since $\sigma(T) = 0$.

*Example.* We use the main theorem to compute the signature of any complex Grassmannian $G_k(n)$, consisting of all k-planes in $\mathbb{C}^{n+k}$; $G_k(n) = GL(n+k, \mathbb{C})/P$ for suitable P, where $W = S_{n+k}$ and $W_P = S_k \times S_n$. (Here S denotes the symmetric group.) One checks that $S_k \times S_n$ is its own normalizer unless $k = n$, in which case $N(W_P)/W_P = Z/2\mathbb{Z}$. One also checks that $D \in S_{n+k}$ is the permutation

$$(1, n+k)(2, n+k-1) \cdots \left(\left[\frac{n+k}{2}\right], \left[\frac{n+k+1}{2}\right]\right).$$

Thus, the conjugates of $W_P$ containing D are easily seen to be in bicorrespondence with those k-element sets S in $\{1, 2, \cdots, n+k\}$ such that $S^D = S$. The number of such sets S is then quite easily calculated to be 0 if k and n - k are both odd, and $\binom{\left[\frac{n+k}{2}\right]}{[k/2]}$ otherwise. We conclude that

$$\sigma(G_k(n)) = \begin{cases} 0 & \text{if } n \text{ and } k \text{ are odd;} \\ 2\dbinom{n}{[k/2]} & \text{if } n = k \text{ even;} \\ \dbinom{\left[\frac{n+k}{2}\right]}{[k/2]} & \text{otherwise.} \end{cases}$$

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Department of Mathematics
University of Notre Dame
South Bend, Indiana 46556