# PROOF OF THE POINCARÉ-BIRKHOFF FIXED POINT THEOREM

M. Brown and W. D. Neumann

## 1. INTRODUCTION

The Poincaré-Birkhoff fixed point theorem (also called Poincaré's last geometric theorem) asserts the existence of at least two fixed points for a so-called area-preserving twist homeomorphism of the annulus. It was formulated as a conjecture and proved in special cases by Poincaré [3], shortly before his death. In 1913 George Birkhoff [1] published a proof which, though correct for one fixed point, overlooked the possibility that this fixed point might have index 0 in deducing the existence of a second fixed point. This error was corrected in his paper [2] of 1925, in which a generalization of the theorem in question is proven, with "area-preserving" replaced by a purely topological condition and "homeomorphism" replaced by a more general situation. However, some mathematicians have claimed that this proof too is incorrect, and the last few years have seen some extensive efforts to try to find a correct proof for the second fixed point.

We present here an elementary proof for two fixed points which is a simple modification of Birkhoff's well known original proof for one fixed point. Our modification to get the second fixed point is essentially the same modification that Birkhoff sketches in the 1925 proof of his topological version to get from one fixed point to two.

This paper is therefore in a sense an expository paper, and to make the proof as transparent as possible we shall restrict to the simplest situation—a twist homeomorphism of the annulus which is just a rotation by a fixed angle on each boundary circle. As we point out in a final section, the proof goes through almost word for word without this restriction. It also extends to more general measures than the standard Lebesgue measure on the annulus.

Since our proof is so close to Birkhoff's proof, which has met with some skepticism, we have felt it advisable to give somewhat more detail than would otherwise be necessary. This is also in keeping with the view of this paper as an expository one.

## 2. STATEMENT OF THE THEOREM

Let $A = \{P \in \mathbb{R}^2 : 1 \leq \|P\| \leq 2\}$ be the annulus. In the literature, a homeomorphism $g: A \to A$ is usually called a "twist homeomorphism" if it "rotates the two boundary components of A in opposite angular directions". This is ambiguous; an anticlockwise rotation by $\theta$ is the same as a clockwise rotation by $(2\pi - \theta)$. The usual way to resolve this ambiguity is by going to the universal cover of A: a homeomorphism $g: A \to A$ is a *twist homeomorphism* if it has no fixed points on $\partial A$ and if it can be lifted to a homeomorphism $\tilde{g}: \tilde{A} \to \tilde{A}$ of the universal cover $\tilde{A} \cong S = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$ of A which moves the two boundary components of $\tilde{A}$ in opposite directions.

Received November 29, 1976. Revision received June 1, 1977.

Michigan Math. J. 24 (1977).

Regarding $\tilde{g}$ as a homeomorphism h of S and extending h in a trivial way to all of $\mathbb{R}^2$, we can formulate the theorem we shall prove as follows.

THEOREM. *Let* $h: \mathbb{R}^2 \to \mathbb{R}^2$ *be an area preserving homeomorphism satisfying*

$$h(x, y) = (x - r_1, y), \qquad y \geq 1;$$

$$h(x, y) = (x + r_2, y), \qquad y \geq 0;$$

$$h(x + 2\pi, y) = h(x, y) + (2\pi, 0),$$

*for some* $r_1, r_2 > 0$. *Then* h *has two distinct fixed points* $F_1$ *and* $F_2$ *which are not in the same periodic family; that is,* $F_1 - F_2$ *is not an integer multiple of* $(2\pi, 0)$.

*Remark.* The periodicity condition $h(x + 2\pi, y) = h(x, y) + (2\pi, 0)$ is precisely the condition that the restriction of h to $S = \{(x, y): 0 \leq y \leq 1\}$ is the lift of a map $g: A \to A$, via the covering map $\pi: S \to A$, $\pi(x, y) = ((y + 1)\cos x, (y + 1)\sin x)$.

Since we are considering the standard area measure dxdy in $\mathbb{R}^2$, it might seem that we are considering a nonstandard area in A; namely, $dr d\theta$ instead of $r dr\, d\theta$. But the homeomorphism $\phi: A \to A$ given in polar coordinates by $\phi(r, \theta) = ((r^2 + 2)/3, \theta)$ takes the one measure to a constant multiple of the other.

To give the proof of the theorem we must first say a few words about rotation numbers and index. This is the only real topology which enters the proof. It uses nothing but the most elementary properties of coverings.

## 3. ROTATION NUMBERS AND INDEX

*Terminology.* If P and Q are distinct points of $\mathbb{R}^2$, then the *direction from* P *to* Q means $D(P, Q) = (Q - P)/\|Q - P\|$.

Let $X \subset \mathbb{R}^2$ be a subset and $h: X \to \mathbb{R}^2$ a homeomorphism of X into $\mathbb{R}^2$ with no fixed points, that is $h(P) \neq P$ for all $P \in X$. If C is any curve in X we want the *index of* C *with respect to* h to be the total rotation that the direction D(P, h(P)) performs as P moves along the curve C. For example, in Figure 1 this direction makes a total of 1 and 1/2 turns in the clockwise (*i.e.*, negative) direction, so the index is -(1 and 1/2).

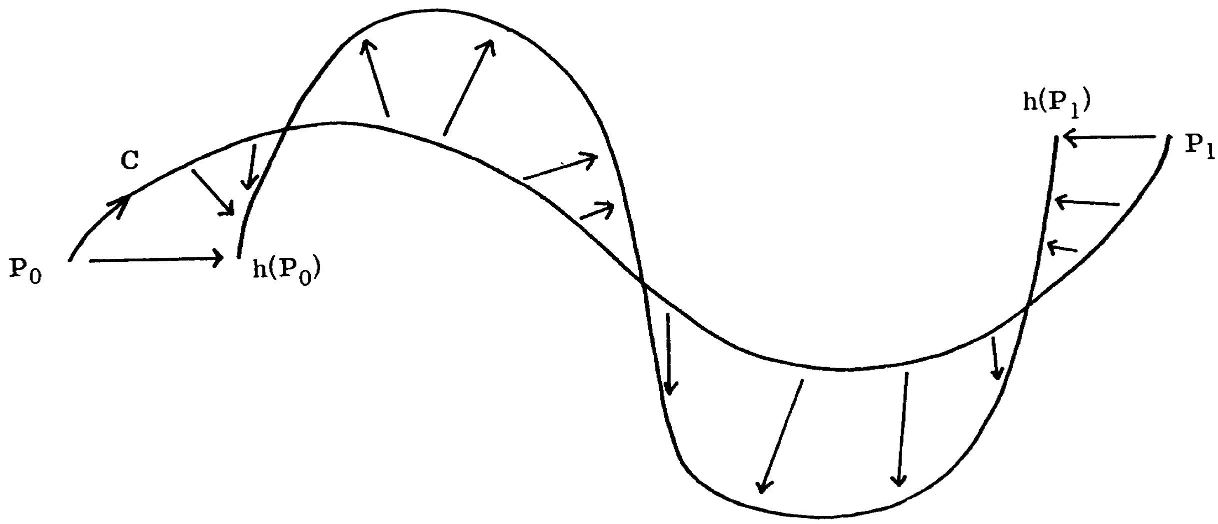



Figure 1

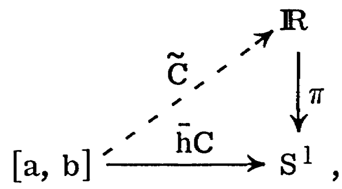
To give a precise definition, we first define a new map

$$\bar{h}\colon X \to S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

by

$$\bar{h}(P) = D(P, h(P)) = (h(P) - P)/\|h(P) - P\| .$$

Then for any curve $C\colon [a, b] \to X$, since $[a, b]$ is simply connected, we can lift the composed map $\bar{h}C\colon [a, b] \to S^1$ to the universal cover of $S^1$:




where $\pi$ is the covering map $\pi(r) = (\cos(r), \sin(r))$. The lifting $\tilde{C}$ is unique up to covering transformations of $\mathbb{R}$; that is, up to addition of integer multiples of $2\pi$. Hence $\tilde{C}(b) - \tilde{C}(a)$ is independent of the lifting and we can define

$$\mathrm{Ind}_h C = (\tilde{C}(b) - \tilde{C}(a))/2\pi .$$

*Properties of index.*

1. *For a one-parameter continuous family of curves* $C$ *or homeomorphisms* $h$ *as above,* $\mathrm{Ind}_h C$ *varies continuously with the parameter.*

2. *If* $C$ *runs from a point* $A$ *to a point* $B$, *then* $\mathrm{Ind}_h C$ *is congruent modulo* 1 *to* $(1/2\pi)$ *times the angle between the directions* $D(A, h(A))$ *and* $D(B, h(B))$.

3. *If* $C = C_1 C_2$ *consists of* $C_1$ *and* $C_2$ *laid end to end (that is,* $C_1 = C|[a, c]$ *and* $C_2 = C|[c, b]$ *with* $a < c < b$), *then* $\mathrm{Ind}_h C = \mathrm{Ind}_h C_1 + \mathrm{Ind}_h C_2$. *If* $-C$ *denotes* $C$ *traversed in the reverse direction, then* $\mathrm{Ind}_h(-C) = -\mathrm{Ind}_h C$.

4. $\mathrm{Ind}_h C = \mathrm{Ind}_{h^{-1}}(h(C))$.

The first property is a simple application of the homotopy lifting property for coverings: any homotopy of the map $\bar{h}C\colon [a, b] \to S^1$ lifts to a homotopy of $\tilde{C}$. Properties 2 and 3 are trivial. Property 4 is an easy calculation—in fact, to calculate $\mathrm{Ind}_{h^{-1}}(h(C))$ we look at the rotation of the direction $D(h(P), P)$. This is $\pi$ plus the direction $D(P, h(P))$ used to calculate $\mathrm{Ind}_h C$, and it rotates the same way the same amount.

Finally, for future reference we remind the reader that a relative version of the homotopy lifting property mentioned above holds: any homotopy of the map $\bar{h}C$ which fixes the endpoints $\bar{h}C(a)$ and $\bar{h}C(b)$, lifts to a homotopy of $\tilde{C}$ which fixes $\tilde{C}(a)$ and $\tilde{C}(b)$. Thus to calculate $\mathrm{Ind}_h C$, we are permitted to perform a homotopy first on $\bar{h}C$ to simplify it, so long as we hold the endpoints fixed.

## 4. PROOF OF THE THEOREM

First some notation: let

$$H_+ = \{(x, y) \in \mathbb{R}^2 : y \geq 1\},$$

$$H_- = \{(x, y) \in \mathbb{R}^2 : y \leq 0\},$$

$$S = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}.$$

Now suppose $h: \mathbb{R}^2 \to \mathbb{R}^2$ is as in the theorem; that is, $h$ is area-preserving, and for some $r_1, r_2 > 0$,

$$h(x, y) = (x - r_1, y), \qquad (x, y) \in H_+;$$

$$h(x, y) = (x - r_2, y), \qquad (x, y) \in H_-;$$

$$h(x + 2\pi, y) = h(x, y) + (2\pi, 0).$$

If $F$ is a fixed point of $h$, then so are all its periodic images $F + (2\pi k, 0)$, $k \in \mathbb{Z}$. We suppose $h$ has at most one such periodic family of fixed points and deduce a contradiction as follows. We shall construct two curves $C$ and $C'$ running from $H_-$ to $H_+$ and avoiding all fixed points of $h$, such that $\mathrm{Ind}_h C = 1/2 = -\mathrm{Ind}_h C'$. This will contradict the following lemma.

LEMMA. *For any curve* $C$ *running from* $H_-$ *to* $H_+$ *and not passing through any fixed point of* $h$,

(a) $\mathrm{Ind}_h C \equiv 1/2 \pmod 1$;

(b) $\mathrm{Ind}_h C$ *is independent of* $C$.

*Proof.* Part (a) is clear by Property 2 of the index. To prove (b), let $\mathrm{Fix}(h)$ be the fixed point set of $h$, so either $\mathrm{Fix}(h) = \emptyset$ or $\mathrm{Fix}(h) = \{F + (2\pi k, 0): k \in \mathbb{Z}\}$. Suppose further that $C_i$, running from $A_i \in H_-$ to $B_i \in H_+$, $i = 1, 2$, are two curves in $\mathbb{R}^2 - \mathrm{Fix}(h)$. Pick any curve $C_3$ from $B_1$ to $B_2$ in $H_+$ and any curve $C_4$ from $A_2$ to $A_1$ in $H_-$, and let $C'$ be the closed curve $C_1 C_3 (-C_2) C_4$ (Figure 2). Since $D(P, h(P))$ is constant in $H_+$ and in $H_-$, $\mathrm{Ind}_h C' = \mathrm{Ind}_h C_1 - \mathrm{Ind}_h C_2$. Thus to show that $\mathrm{Ind}_h C_1 = \mathrm{Ind}_h C_2$, we must show $\mathrm{Ind}_h C' = 0$. But the fundamental group $\pi_1(\mathbb{R}^2 - \mathrm{Fix}(h), A_1)$ is generated by paths which start from $A_1$, run along a curve $C_0$ to near a fixed point (if there are any), loops around this fixed point, and returns by $-C_0$ to $A_1$. Hence $C'$ is deformable into a composition of such paths and it is sufficient to show that $\mathrm{Ind}_h$ is zero for any such path (see Figure 3). But any such path encircling a single fixed point can be deformed into a path $C''$ as in Figure 4. The contributions to $\mathrm{Ind}_h C''$ of the top and bottom horizontal sections are both zero, while the contributions from the two vertical sections cancel exactly, by the periodicity of $h$. Thus $\mathrm{Ind}_h C'' = 0$, and the lemma is proved. The reader may have realized that we have just proved a special case of the Lefschetz fixed point theorem for the annulus.

We shall now assume for convenience that the periodic family of fixed points, if it occurs, lies on the lines $x \equiv 0 \pmod{2\pi}$. This can be achieved by a trivial change of coordinates.

Let $W$ denote the periodic union of vertical strips

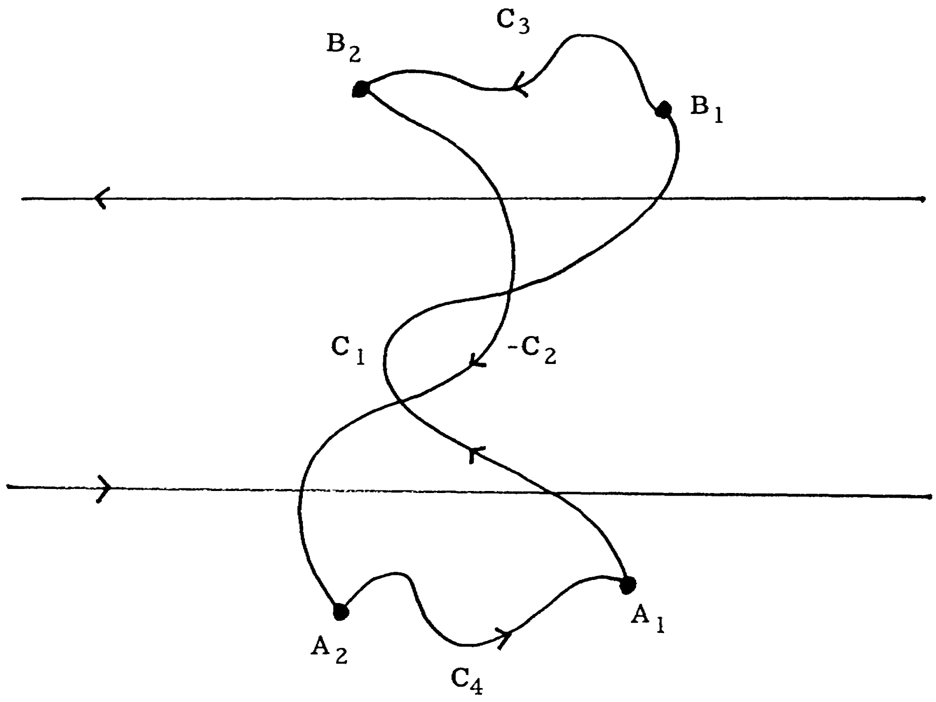



Figure 2

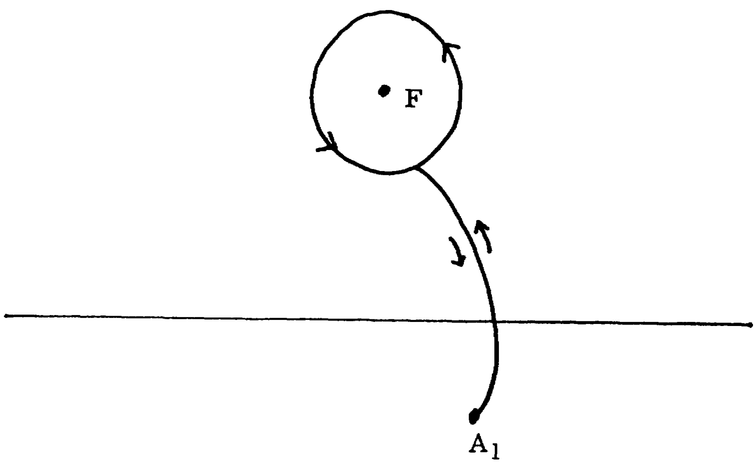



Figure 3

$$W = \{(x, y): 2k\pi + \pi/2 \leq x \leq 2k\pi + 3\pi/2 \text{ for some } k \in \mathbb{Z}\},$$

so W does not contain any of the fixed points. Choose $\varepsilon > 0$ such that

$$\|P - h(P)\| > \varepsilon \quad \text{for all } P \in W.$$

This is possible: it clearly suffices by periodicity to satisfy this only for P in the compact region $\{(x, y): \pi/2 \leq x \leq 3\pi/2,\ 0 \leq y \leq 1\}$, and on this region the function $\phi(P) = \|P - h(P)\|$ is continuous and positive and hence has a positive minimum.

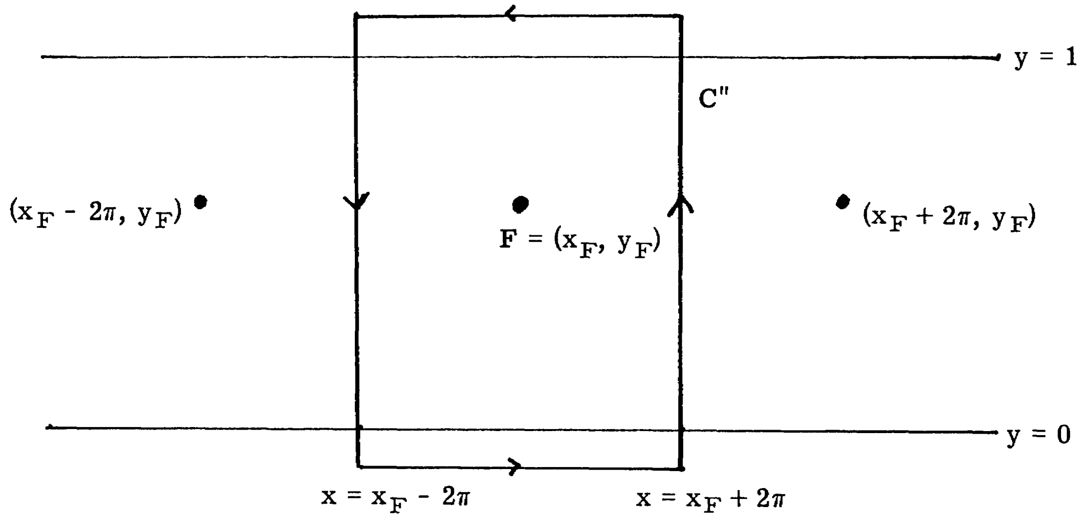



Figure 4

Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x, y + (\varepsilon/2)(|\cos(x)| - \cos(x)))$. Observe that T is just a constant upward shift on any vertical line (Figure 5), so by elementary calculus it is area-preserving. Furthermore T moves only points of W, and it moves each a distance at most $\varepsilon$. Hence Th has no fixed point in W, since $P = Th(P)$ with $P \in W$ would imply

$$\varepsilon \geq \|Th(P) - h(P)\| = \|P - h(P)\| > \varepsilon .$$

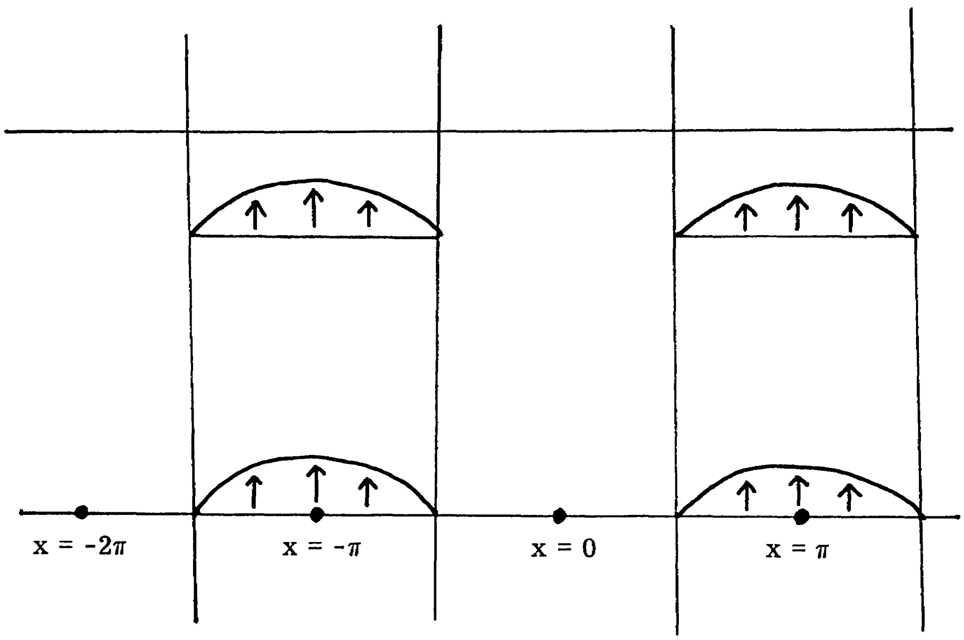



Figure 5

Let us now summarize the course of the remaining argument. We shall show that there exists a point $P_0 \in H_-$ such that $(Th)^n P_0 \in H_+$ for some n. This will allow us to find a curve C from $H_-$ to $H_+$ which is a "flow line" for Th; that is, it is mapped into itself (except near one endpoint) by Th. However, the index can be easily calculated for a flow line, and we shall see that $\mathrm{Ind}_{Th} C$ is very close to 1/2. Since Th is close to h, we shall be able to deduce that $\mathrm{Ind}_h C = 1/2$. An easy symmetry argument then shows that another curve C' exists with $\mathrm{Ind}_h C' = -1/2$, giving the required contradiction.

To see that a point $P_0$ as above exists, we proceed as follows. Define

$$D_0 = H_- - (Th)^{-1} H_-, \qquad D_1 = (Th) D_0 = (Th) H_- - H_-,$$

and more generally

$$D_i = (Th)^i D_0 \qquad \text{for } i \in \mathbb{Z}.$$

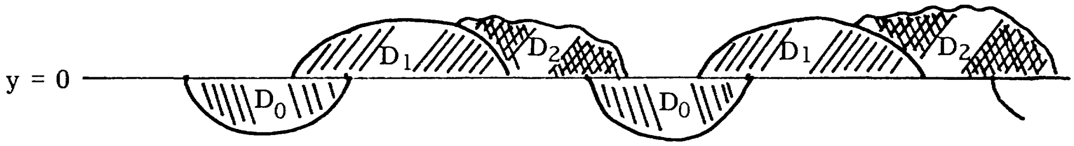



Figure 6

Since $D_1 \subset \mathrm{Int}(S) \cup H_+$ and $Th(\mathrm{Int}(S) \cup H_+) \subset \mathrm{Int}(S) \cup H_+$, a trivial induction shows $D_i \subset \mathrm{Int}(S) \cup H_+$ for all $i \geq 1$. Similarly, $D_i \subset H_-$ for all $i \leq 0$. In particular, $D_i \cap D_0 = \emptyset$ for $i > 0$, so by applying powers of $Th$ to this equation we see that $D_j \cap D_k = \emptyset$ whenever $j \neq k$.

Now by "area" we shall mean area in the "rolled up" plane

$$S^1 \times \mathbb{R} = \mathbb{R}^2/((x, y) \equiv (x + 2\pi, y)).$$

In this sense each $D_i$ has the same area (namely, $2\varepsilon$), since $T$ and $h$ preserve area, and hence $Th$ does also. Thus the $D_i$ with $i \geq 1$ must eventually exhaust $S$, which has area $2\pi$, and hence eventually intersect $H_+$.

Since $D_n \subset (Th)^n H_-$, we have shown that there exists an $n > 0$ such that $(Th)^n H_- \cap H_+ \neq \emptyset$. For such an $n$ choose a point $P_n \in (Th)^n H_-$ with maximal $y$-coordinate. Such a $P_n$ need not be unique, but it exists, since by periodicity, we need only look at the compact region $Th^n(H_-) \cap \{(x, y)\colon y \geq 0,\ 0 \leq x \leq 2\pi\}$, and $y$ is a continuous function on this region, so it attains a maximum. Define

$$P_i = (x_i, y_i) = (Th)^{i-n} P_n \qquad \text{for } i \in \mathbb{Z},$$

so $P_{i+1} = Th(P_i)$ for all $i$, and $P_{-1}, P_0 \in H_-$ and $P_n, P_{n+1} \in H_+$. Let $C_0$ be the straight line segment from $P_{-1}$ to $P_0$, and let $C_i = (Th)^i C_0$ for $i \in \mathbb{Z}$. $P_0$ actually lies on the line $y = 0$, but we do not need this (Figure 7). Let $C = C_0 C_1 \cdots C_{n-1} C_n$, so $Th(C) = C_1 C_2 \cdots C_n C_{n+1}$. We shall need the following facts about $C$.

*Properties of* $C$

1. *The curve* $CC_{n+1} = C_0 \cdots C_{n+1}$ *has no double points.*
2. *No point of* $C$ *has larger* $y$*-coordinate than* $P_{n+1}$.
3. *No point of* $Th(C)$ *has smaller* $y$*-coordinate than* $P_{-1}$.

Property 2 is clear by observing that $C \subset (Th)^n H_-$ and that for $(x, y) \in H_-$ we have $y \leq y_n \leq y_{n+1}$, by the choice of $P_n$. Property 3 is proved by an easy induction using the observation that if $Th(x, y) = (x', y')$ and $y \geq y_{-1}$, then $y' \geq y_{-1}$. To see Property 1, suppose two curves $C_i$ and $C_j$ with $i \neq j$ intersected in some point other than the common endpoint which occurs when $|i - j| = 1$. By applying a large negative power of $Th$, we can make $i$ and $j$ negative. But this would be absurd, since for $i$ nonpositive $C_i$ lies completely in the strip

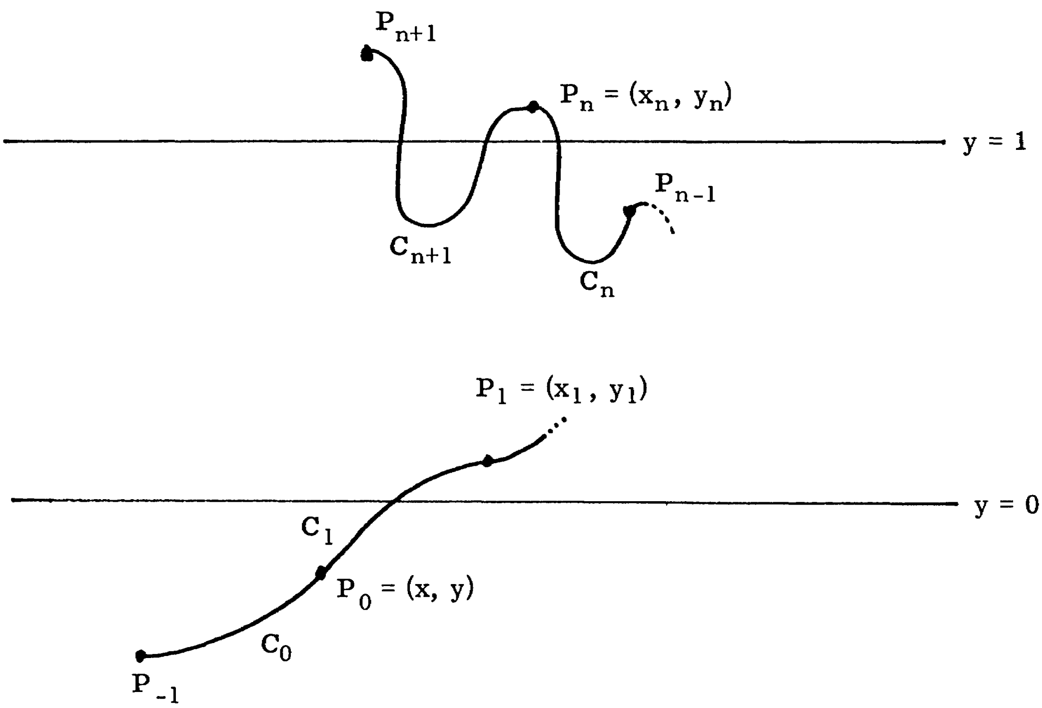



Figure 7

$$\{(x, y);\ x_0 + (i - 1)r_2 \leq x \leq x_0 + ir_2\}$$

and intersects the boundaries of this strip only in its endpoints (because this is true for $C_0$, and the x-component of $(Th)^{-1}$ in $H_-$ is just translation by $-r_2$). For $i \neq j$ these strips intersect at most in a boundary.

We can now calculate $\mathrm{Ind}_{Th}C$. By construction,

$$P_{n+1} = (x_{n+1}, y_{n+1}) = (x_n - r_1, y_n + \delta), \qquad 0 \leq \delta_1 \leq \varepsilon;$$

$$P_0 = (x_0, y_0) = (x_{-1} + r_2, y_{-1} + \delta_2), \qquad 0 \leq \delta_2 \leq \varepsilon.$$

Thus the angle between $D(P_{-1}, P_0)$ and $D(P_n, P_{n+1})$ is

$$\theta = \pi - (\arctan(\delta_1/r_1) + \arctan(\delta_2/r_2))$$

(see Figure 8), whence

$$\mathrm{Ind}_{Th}C \equiv \theta/2\pi = 1/2 - (1/2\pi)(\arctan(\delta_1/r_1) + \arctan(\delta_2/r_2)) \pmod 1.$$

Observe that $0 \leq \delta_i \leq \varepsilon < r_i$ for $i = 1, 2$, so $0 \leq \arctan(\delta_i/r_i) < \pi/4$, so $1/4 < \theta/2\pi \leq 1/2$. We shall show that the above congruence can be replaced by equality; that is, $\mathrm{Ind}_{Th}C = \theta/2\pi$. Here is an intuitive sketch of the argument.

$\mathrm{Ind}_{Th}C$ is the total rotation of $D(P, Q)$, $Q = Th(P)$, as $P$ moves along $C$. The idea is to observe that we get the same total rotation if we first hold $P = P_{-1}$ fixed and just move $Q$ along $Th(C)$ from $Th(P_{-1}) = P_0$ to $P_{n+1}$, and then hold $Q$ at $P_{n+1}$ and move $P$ along $C$ from $P_{-1}$ to $P_n$. But now instead of moving $Q$ along $Th(C)$ and $P$ along $C$, we can move $Q$ along the straight line segment from $P_0$ to $P_{n+1}$ and then $P$ along the straight line segment from $P_{-1}$ to $P_n$. In this situation the total rotation is easily seen to be as claimed (see Figure 8).

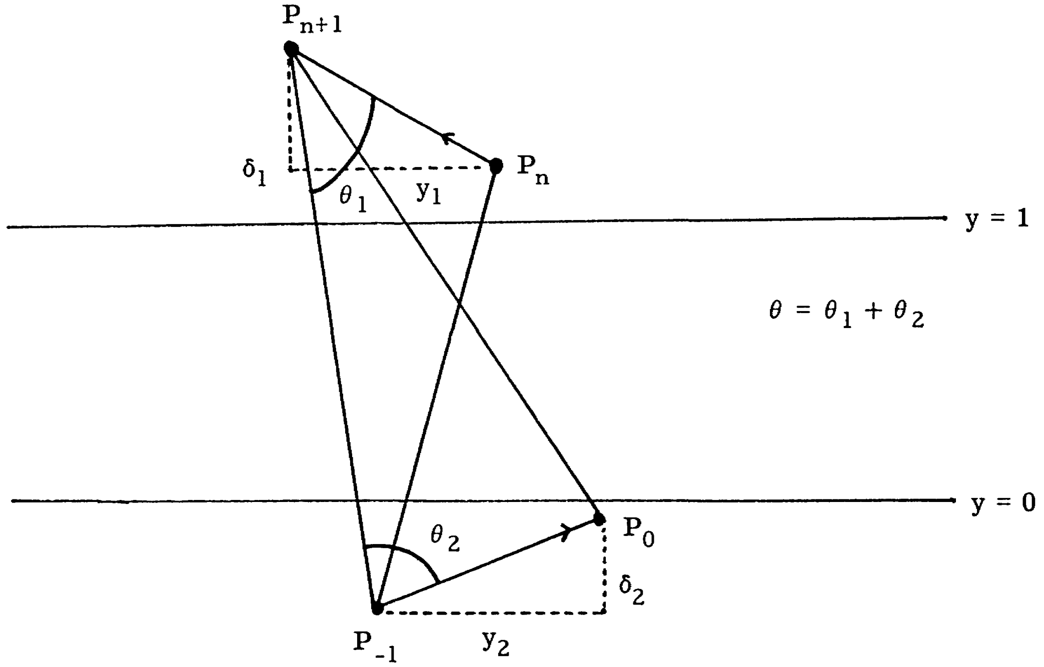



Figure 8

For the precise argument, parametrize the curve $C_0$ by a map $[-1, 0] \to \mathbb{R}^2$ and then extend this to a parametrization $P: [-1, n+1] \to \mathbb{R}^2$ of $CC_{n+1}$ by requiring $Th(P(t)) = P(t+1)$ for $-1 \le t \le n$. Thus $P(i) = P_i$ for $i = -1, 0, \cdots, n+1$.

By definition, $\mathrm{Ind}_{Th} C$ is calculated from

$$\overline{P}: [-1, n] \to S^1, \qquad \overline{P}(t) = D(P(t), P(t+1)).$$

We can use instead $\overline{P}_0: [-1, 2n+1] \to S^1$ defined by

$$\overline{P}_0 = \begin{cases} \overline{P}(t), & -1 \le t \le n, \\ \overline{P}(n), & n \le t \le 2n+1. \end{cases}$$

We now define a continuous family of maps $\overline{P}_\lambda: [-1, 2n+1] \to S^1$, $0 \le \lambda \le n+2$, all with the same endpoints, such that the final map $\overline{P}_{n+2}$ is just a monotone angle increase from $\arctan(\delta_2/r_2)$ through $\pi - \arctan(\delta_1/r_1)$. Our claim then follows immediately from the remark on homotopy lifting at the end of Section 3.

We define the family $\overline{P}_\lambda$ in two parts.

*Part* 1: $0 \le \lambda \le n+1$.

$$\overline{P}_\lambda(t) = \begin{cases} D(P(-1), P(t+1)), & -1 \le t \le \lambda - 1; \\ D(P(t-\lambda), P(t+1)), & \lambda - 1 \le t \le n; \\ D(P(t-\lambda), P(n+1)), & n \le t \le n+\lambda; \\ D(P(n), P(n+1)), & n+\lambda \le t \le 2n+1. \end{cases}$$

Observe that for any $\lambda$ and t as above, $\overline{P}_\lambda(t)$ always has the form $D(P(t_0), P(t_1))$ with $-1 \leq t_0 < t_1 \leq n+1$, and is hence well defined, since $CC_{n+1}$ is a simple curve (Property 1 of C), so $P(t_0) \neq P(t_1)$.

*Part* 2. $n+1 \leq \lambda \leq n+2$. Let $P'\colon [0, n+1] \to \mathbb{R}^2$ and $P''\colon [-1, n] \to \mathbb{R}^2$ be the straight line segments from P(0) to P(n + 1) and from P(-1) to P(n), respectively. For $0 \leq \mu \leq 1$, define

$$\overline{P}_{n+1+\mu}(t) = \begin{cases} D(P(-1), (1-\mu)P(t+1) + \mu P'(t+1)), & -1 \leq t \leq n; \\ D((1-\mu)P(t-n-1) + \mu P''(t-n-1), P(n+1)), & n \leq t \leq 2n+1. \end{cases}$$

To see that this is well defined, we must check that the right-hand side always has the form D(P, Q) with $P \neq Q$. For $P = (1-\mu)P(t-n-1) + \mu P''(t-n-1)$ and $Q = P(n+1)$, $n \leq t \leq 2n+1$, P always has smaller y-coordinate than Q, by Property 2 of C, except possibly when $t = 2n+1$ or $\mu = 0$, in which case

$$P = P(t-n-1) \neq P(n+1) = Q,$$

by Property 1 of C. Similarly, $P = P(-1)$ never equals

$$Q = (1-\mu)P(t+1) + \mu P'(t+1) \qquad \text{for } -1 \leq t \leq n,$$

by Properties 3 and 1 of C.

It is now a trivial trigonometric calculation to see that $\overline{P}_{n+2}(t)$, $-1 \leq t \leq 2n+1$, is a monotone angle increase by exactly $\theta$, as claimed, whence

$$\mathrm{Ind}_{Th}\, C = \theta/2\pi = 1/2 - (1/2\pi)(\arctan(\delta_1/r_1) + \arctan(\delta_2/r_2)).$$

Thus we have completed the calculation of $\mathrm{Ind}_{Th}\, C$.

Now define $T_s\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T_s(x, y) = (x, y + (s\varepsilon/2)(|\cos(x)| - \cos(x)))$, so $T_0 = \mathrm{id}$ and $T_1 = T$. Then for $0 \leq s \leq 1$, $\mathrm{Ind}_{T_s h}\, C$ is defined and

$$\mathrm{Ind}_{T_s h}\, C \equiv 1/2 - (1/2\pi)(\arctan(s\delta_1/r_1) + \arctan(s\delta_2/r_2)) \pmod 1,$$

for the same reasons that the corresponding statement held for $Th = T_1 h$. But we have seen that this congruence is actually an equality for $s = 1$, so by continuity of Ind, it is an equality for all $0 \leq s \leq 1$. In particular, for $s = 0$ we get $\mathrm{Ind}_h C = 1/2$, as desired.

We can now repeat the whole argument using $h^{-1}$ in place of h but using the same T. Everything is then left-right reversed and we get a curve $C_0$ with $\mathrm{Ind}_{h^{-1}}\, C_0 = -1/2$. Property 4 of Ind then shows for $C' = h^{-1} C_0$ that $\mathrm{Ind}_h C' = -1/2$. We have thus found the curves C and C' giving the desired contradiction to the lemma.

## 5. FINAL REMARKS

In the proof we did not really need that h be an actual translation on each boundary of S; it suffices that h move the boundaries of S in opposite directions. For if we extend h in the natural way to all $\mathbb{R}^2$, it is then no longer area-preserving on $H_+$ and $H_-$. This, however, does not matter since the proof needs only that h be area-preserving in S.

The proof also easily extends to measures other than the standard area measure on the annulus. However, the best result we know in this direction, which cannot be obtained so directly, seems to be Birkhoff's topological version which replaces "area-preserving" by the condition: there is no open neighborhood U of one of the boundaries of the annulus such that U is contained in h(U) as a nondense subset. Birkhoff's proof follows similar lines to what we have presented here, but the construction of the "approximate flow lines" C and C' for h, connecting the inside and outside of the annulus and having indices +1/2 and -1/2, is much more delicate.

Using a slight extension of the argument described, one can obtain a quite precise lower bound on the number of components of the set of periodic points of period exactly n of h: A → A, generalizing the Poincaré-Birkhoff theorem. This bound is asymptotic to a constant times the Euler function $\phi(n)$. Details will appear in a paper by the second author.

## REFERENCES

1. G. D. Birkhoff, *Proof of Poincaré's geometric theorem.* Trans. Amer. Math. Soc. 14 (1913), 14-22.

2. ———, *An extension of Poincaré's last geometric theorem.* Acta Math. 47 (1925), 297-311.

3. H. Poincaré, *Sur un théorème de géométrie.* Rend. Circ. Mat. Palermo 33 (1912), 375-407.

Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109
and
Department of Mathematics
University of Maryland
College Park, Maryland 20742