FOURIER-STIELTJES TRANSFORMS OF STRONGLY CONTINUOUS MEASURES

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1. INTRODUCTION AND PRELIMINARIES

Let G denote throughout a compact abelian group and Γ its dual. For μ belonging to M(G), the space of regular bounded Borel measures on G, the Fourier-Stieltjes transform of μ is given by $\hat{\mu}(\gamma) = \int_G \overline{\gamma(g)}\, d\mu(g)$, $\gamma \in \Gamma$.

Let $0 < \varepsilon < 1$. We shall be concerned with measures μ whose transforms satisfy the following separation condition:

(1, ε) For every $\gamma \in \Gamma$, either $|\hat{\mu}(\gamma)| \geq 1$ or $|\hat{\mu}(\gamma)| < \varepsilon$.

We shall call μ *strongly continuous* if

(2) $|\mu|(g + H) = 0$ for all $g \in G$ and all closed subgroups H of G such that G/H is infinite.

The main result of this paper may be stated qualitatively as follows. For each $C > 0$, there is some $\varepsilon = \varepsilon(C) > 0$ such that if μ satisfies (1, ε), (2), and $\|\mu\| \leq C$, then $\Lambda = \{\gamma \in \Gamma\colon |\hat{\mu}(\gamma)| \geq 1\}$ is a finite set. It will turn out that $\varepsilon = \varepsilon(C)$ may be chosen independently of G. An alternative formulation of our main result is the following. There is some constant A independent of G such that $\|\mu\| \geq -A(\log \varepsilon)^{1/5}$ for all $\mu \in M(G)$ satisfying (1, ε) and (2), if Λ is infinite.

Previous versions of this theorem include de Leeuw's and Katznelson's [2, p. 221] for the case G = T (the circle group). Ramsey [5] has proved the theorem for those Γ whose torsion subgroup is finite. He also obtained quantitative bounds on the size of Λ. In the general case treated here, such bounds do not exist. To see that, let G be the familiar Cantor group $\prod (\mathbb{Z}/2\mathbb{Z})$, and let Λ be a finite subgroup of Γ of order 2^n. Define μ to be the trigonometric polynomial $\sum_{\gamma \in \Lambda} \gamma(g)$ on G. Since μ is the normalized Haar measure on the compact subgroup $\Lambda^{\perp}$ of G, we have that $\|\mu\| = 1$. It is clear that μ satisfies (1, ε) for every $\varepsilon > 0$ and (2); however, the order of Λ can be arbitrarily large.

An alternate expression of (2) is possible using the canonical homomorphism ϕ of G onto G/H. Define $\phi(\mu)$ to be that measure in M(G/H) determined by the equation $\phi(\mu)(B) = \mu(\phi^{-1}(B))$ for all Borel subsets B of G/H. The equation

$$\int_{G/H} f \, d(\phi(\mu)) = \int_G f \circ \phi \, d\mu$$

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holds for all $f \in C(G/H)$, and from this it follows that $\phi(\mu)^\wedge = \hat{\mu}|_{H^\perp}$. The strong continuity of μ is equivalent to the continuity of $\phi(|\mu|)$ for all closed subgroups H whose index in G is infinite. Since a given Haar-measurable function on G may be approximated in measure by trigonometric polynomials, it follows that the strong continuity of μ is equivalent to the continuity of $\phi(\gamma\mu)$ for all closed subgroups H whose index in G is infinite and for all $\gamma \in \Gamma$.

If $\mu \in M(G)$, then $\mu^\sim$ denotes the measure defined by the equation $\mu^\sim(B) = \overline{\mu(-B)}$ for all Borel subsets B of G. For a bounded Borel function h on G, $h\mu$ denotes that measure in M(G) satisfying $\int_G f\, d(h\mu) = \int_G fh\, d\mu$, $f \in C(G)$. If Λ is a subset of Γ, then $\overline{\Lambda}\mu$ denotes the set $\{\overline{\lambda}\mu: \lambda \in \Lambda\}$. Note that $(\overline{\lambda}\mu)^\wedge(\gamma) = \hat{\mu}(\lambda + \gamma)$ for all $\gamma \in \Gamma$.

The following is a characterization of strong continuity in terms of the Fourier-Stieltjes transform.

LEMMA 1. *Let* $\mu \in M(G)$. *Then* μ *is strongly continuous if and only if for every infinite subgroup* X *of* Γ *and* $\gamma \in \Gamma$, 0 *is the unique constant function in the weak closure of the convex hull of the set of translates by elements of* X *of* $|\phi(\gamma\mu)^\wedge|^2$. ($\phi$ denotes the canonical projection from M(G) onto $M(G/X^\perp)$.)

Proof. Suppose that μ is strongly continuous. Equivalently, suppose that $\phi(\gamma\mu)$ is a continuous measure on G/H for every closed subgroup H of infinite index in G and every $\gamma \in \Gamma$. By Wiener's theorem [6, p. 118], 0 belongs to the weak closure of the convex hull of the set of translates of $|\phi(\gamma\mu)^\wedge|^2$ by elements of $H^\perp$. Since $|\phi(\gamma\mu)^\wedge|^2$ is the Fourier-Stieltjes transform of a measure on G/H, it is a weakly almost periodic function on $H^\perp$ (see [3], Theorem 11.2). By a theorem of Eberlein ([3], Theorem 5.3), 0 is the only constant function in the weak closure of the convex hull of the set of translates of $|\phi(\gamma\mu)^\wedge|^2$.

Conversely, if for all infinite subgroups X of Γ, 0 is the only constant function in the weak closure of the convex hull of the translates of $|\phi(\gamma\mu)^\wedge|^2$ (here ϕ: $M(G) \to M(G/X^\perp)$), then Wiener's theorem implies that $\phi(\gamma\mu)$ is a continuous measure on $G/X^\perp$; *i.e.*, μ is strongly continuous.

The following technical lemma is contained in [5] (see Theorem 1 and the proof of Theorem 2). It is based on an idea of Davenport [1].

LEMMA 2. *Let* G *be a compact abelian group, and let* $\mu \in M(G)$. *Suppose that for an integer* $r > 16$, *there are elements* γ_0, $\{\gamma_{k,j}\}_{j=1}^r$, $1 \le k \le r^2$, *in* Λ *such that if* $P_0 = \{\gamma_0\}$ *and*

$$P_k = P_{k-1} \cup \{\gamma_{k,i}\}_{i=1}^r \cup \left\{ \bigcup_{i<j} P_{k-1} + \gamma_{k,i} - \gamma_{k,j} \right\},$$

we have for $1 \le k \le r^2$,

(3, k) $$(P_{k-1} + \gamma_{k,i} - \gamma_{k,j}) \cap \Lambda = \emptyset, \qquad 1 \le i < j \le r.$$

Set $\varepsilon = 2^{-1} r^{3/2} r^{-2r^2}$. *If* μ *satisfies* (1, ε), *then* $\|\mu\| \ge 4^{-1} r^{1/2}(1 - e^{-2})$.

Proof. Assume on the contrary that $4^{-1} r^{1/2}(1 - e^{-2}) > \|\mu\|$. We define trigonometric polynomials $\phi_0, \cdots, \phi_{r^2}$ inductively as follows:

$$\phi_0(\cdot) = \bar{\sigma}(\hat{\mu}(\gamma_0))(\gamma_0, \cdot),$$

where $\sigma(x) = \text{signum } x = x\,|x|^{-1}$ for $x \neq 0$, and $\bar{\sigma}(x) = \bar{x}\,|x|^{-1}$;

$$\phi_k(\cdot) = \left\{ 1 - 2r^{-2} - r^{-3} \sum_{i<j} \bar{\sigma}(\hat{\mu}(\gamma_{k,i}))\,\sigma(\hat{\mu}(\gamma_{k,j}))\,(\gamma_{k,i} - \gamma_{k,j}, \cdot) \right\} \phi_{k-1}(\cdot)$$

$$+ r^{-5/2} \sum_j \bar{\sigma}(\hat{\mu}(\gamma_{k,j}))\,(\gamma_{k,j}, \cdot).$$

Note that if $P_0, \cdots, P_{r^2}$ are defined as in the statement of Lemma 2, each ϕ_k is a P_k-polynomial. By [1, Lemmas 1 and 2], $|\phi_k(g)| \leq 1$ for all $g \in G$. Let $I_k = \int_G \phi_k(g)\, d\mu(-g)$. Then $I_0 = |\hat{\mu}(\gamma_0)| \geq 1$. Moreover,

$$\Re(I_k) \geq (1 - 2r^{-2})\, \Re(I_{k-1}) + 2^{-1} r^{-3/2}. \tag{4}$$

To verify (4), we write

$$I_k = (1 - 2r^{-2}) I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(\gamma_{k,j})|$$

$$- r^{-3} \sum_{\gamma \in P_{k-1}} \sum_{i<j} \hat{\phi}_{k-1}(\gamma)\, \bar{\sigma}(\hat{\mu}(\gamma_{k,i}))\, \sigma(\hat{\mu}(\gamma_{k,j}))\, \hat{\mu}(\gamma + \gamma_{k,i} - \gamma_{k,j})$$

$$= (1 - 2r^{-2}) I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(\gamma_{k,j})| - r^{-3} A.$$

In view of (1, ε) and (3, k), we observe that each term of A is bounded in modulus by $2^{-1} r^{3/2} r^{-2r^2}$, and that the number of terms in A is at most

$$[r(r - 1)/2]\, \text{card}(P_{k-1}) \leq r^2 \text{card}(P_{k-1}) \leq r^{2k}.$$

Note that $\text{card}(P_0) = 1 \leq r^{2 \cdot 0}$ and that

$$\text{card}(P_{k+1}) = \text{card}\left(P_k \cup \{\gamma_{k+1,j}\}_{j=1}^{r} \cup \left[\bigcup_{i<j} P_k + \gamma_{k+1,i} - \gamma_{k+1,j} \right] \right)$$

$$\leq \text{card}(P_k) + r + [r(r - 1)/2]\text{card}(P_k)$$

$$\leq (1 + r + [r(r - 1)/2])\,\text{card}(P_k) \leq r^2 \text{card}(P_k);$$

hence, $\text{card}(P_k) \leq r^{2k}$.

It follows by induction from (4) that

$$\Re(I_k) \geq 4^{-1} r^{1/2} - (1 - 2r^{-2})^k (4^{-1} r^{1/2} - 1).$$

For $k = r^2$, we conclude that

$$|I_k| \geq \Re(I_k) \geq 4^{-1} r^{1/2} - (1 - 2r^{-2})^{r^2} (4^{-1} r^{1/2} - 1)$$
$$\geq 4^{-1} r^{1/2} - e^{-2}(4^{-1} r^{1/2} - 1) \geq 4^{-1} r^{1/2}(1 - e^{-2}) > \|\mu\|,$$

although $|\phi_k(g)| \leq 1$ for all $g \in G$. This contradiction establishes the lemma.

2. THE MAIN RESULT

THEOREM. *Let* G *be a compact abelian group, and let* $\mu \in M(G)$. *Set* $\varepsilon = 2^{-1} r^{3/2} r^{-2r^2}$, *and suppose that* $r > 16$ *is an integer such that* $4^{-1} r^{1/2}(1 - e^{-2}) > \|\mu\|$. *Then if* μ *satisfies* $(1, \varepsilon)$ *and* (2), *it follows that* Λ *is a finite set.*

Proof. Suppose on the contrary that Λ is infinite.

It will be convenient to begin by reducing to the case of metrizable G. This will allow us to work with sequences instead of nets in the dual group Γ. To this end, select a countably infinite subset of Λ and let Γ' be the (discrete) countable subgroup of Γ generated by that countable set. Let $\phi(\mu)$ be the canonical image of μ in $M(G/\Gamma'^{\perp})$. Since μ satisfies $(1, \varepsilon)$, and since $\phi(\mu)^{\wedge} = \hat{\mu}|_{\Gamma'}$, it follows that $\phi(\mu)$ also satisfies $(1, \varepsilon)$. Suppose that H is a closed subgroup of $G/\Gamma'^{\perp}$ of infinite index; then $\phi^{-1}(H)$ is a closed subgroup of infinite index in G, and therefore $\phi(|\mu|)(g + H) = |\mu|(\phi^{-1}(g + H)) = 0$. This last equality shows that $\phi(\mu)$ also satisfies (2). Since $\|\phi(\mu)\| \leq \|\mu\|$, it clearly follows that $4^{-1} r^{1/2}(1 - e^{-2}) > \|\phi(\mu)\|$. Therefore, we may and shall assume for the remainder of the proof that G is metrizable.

Choose some weak* cluster point ν of $\overline{\Lambda}\mu$ of minimal norm. Since $|(\overline{\lambda}\mu)^{\wedge}(0)| = |\hat{\mu}(\lambda)| \geq 1$ for $\lambda \in \Lambda$, we must have $\nu \neq 0$. Now, since μ satisfies $(1, \varepsilon)$, either $|\hat{\nu}(\gamma)| \leq \varepsilon$ or $|\hat{\nu}(\gamma)| \geq 1$ for all $\gamma \in \Gamma$. Let E denote $\{\gamma \in \Gamma: |\hat{\nu}(\gamma)| \geq 1\}$. In the next paragraph we shall prove that $\overline{E}\nu$ is contained in the weak* closure of $\overline{\Lambda}\mu$. As a consequence, the weak* closure $(\overline{E}\nu)^{-}$ consists of measures all of whose norms are $\|\nu\|$. Hence, the weak* and norm topologies agree on $(\overline{E}\nu)^{-}$ ([4], Lemma 2.1). Therefore, $(\overline{E}\nu)^{-}$ is compact in the norm topology.

Suppose that $\overline{\lambda}_n \mu$ converges weak* to ν, where $\lambda_n \in \Lambda$, and let $\rho \in E$. Then $\overline{\rho\lambda_n}\mu$ converges weak* to $\overline{\rho}\nu$, and $|(\overline{\rho}\nu)^{\wedge}(0)| = |\hat{\nu}(\rho)| \geq 1$. Thus, for large n, we have $|\hat{\mu}(\rho + \lambda_n)| \geq 1$, since μ satisfies $(1, \varepsilon)$ with $\varepsilon < 1$. Thus $\rho + \lambda_n \in \Lambda$, for large n, and $\overline{\rho}\nu$ is a weak* cluster point of $\overline{\Lambda}\mu$.

We now proceed to show as in [4] that E is a finite union of cosets of some subgroup of Γ. We define an equivalence relation on E: $\lambda_1 \sim \lambda_2$ if and only if $-\lambda_1 + E = -\lambda_2 + E$. Note that $\|\overline{\lambda}_1 \nu - \overline{\lambda}_2 \nu\| < 1 - \varepsilon$ implies that $|\hat{\nu}(\lambda_1 + \gamma)| \geq 1$ if and only if $|\hat{\nu}(\lambda_2 + \gamma)| \geq 1$ for $\gamma \in \Gamma$, since either $|\hat{\nu}(\gamma)| \leq \varepsilon$ or $|\hat{\nu}(\gamma)| \geq 1$ for all $\gamma \in \Gamma$; *i.e.*, $\lambda_1 + \gamma \in E$ if and only if $\lambda_2 + \gamma \in E$, so $-\lambda_1 + E = -\lambda_2 + E$. Thus,

$$\|\overline{\lambda}_1 \nu - \overline{\lambda}_2 \nu\| < 1 - \varepsilon$$

implies that $\lambda_1 \sim \lambda_2$. Since $(\overline{E}\nu)^{-}$ is compact in the norm topology (and $\overline{E}\nu$ is dense in that set in the norm topology), some finite number of neighborhoods $U_\lambda = \{\omega \in M(G): \|\omega - \overline{\lambda}\nu\| < 1 - \varepsilon\}$ with $\lambda \in E$ cover $(\overline{E}\nu)^{-}$. Thus, E consists of a finite number of equivalence classes E_i, $1 \leq i \leq m$, under the equivalence relation $\sim$.

We now prove that there is a subgroup X of Γ such that $E_i = \tau_i + X$ for some τ_i. First, let $\tau_i \in E_i$. We show that $-\tau_i + E_i$ is a subgroup of Γ.

Let $\lambda_1, \lambda_2 \in -\tau_i + E_i$. It will be shown that $\lambda_1 - \lambda_2 \in -\tau_i + E_i$. To do this, it suffices to show that $(\tau_i + \lambda_1 - \lambda_2) \sim \tau_i$. Now,

$$-(\tau_i + \lambda_1 - \lambda_2) + E = -\tau_i + (\lambda_2 + \tau_i) - (\lambda_1 + \tau_i) + E$$

$$= -\tau_i + (\lambda_2 + \tau_i) - (\lambda_2 + \tau_i) + E = -\tau_i + E.$$

Since $0 = -\tau_i + \tau_i$ is in $-\tau_i + E_i$, $0 - \lambda_2 = -\lambda_2 \in -\tau_i + E_i$ for all $\lambda_2 \in -\tau_i + E_i$. Hence, $\lambda_1 + \lambda_2 = \lambda_1 - (-\lambda_2) \in -\tau_i + E_i$ for all $\lambda_1, \lambda_2 \in -\tau_i + E_i$. This shows that $-\tau_i + E_i$ is a group.

We now show that $-\tau_i + E_i = -\tau_j + E_j$. Let $\lambda \in -\tau_i + E_i$. We show that $\lambda + \tau_j \sim \tau_j$, which implies that $\lambda \in -\tau_j + E_j$. Write λ as $-\tau_i + \lambda'$ for some $\lambda' \in E_i$. Then

$$-(\lambda + \tau_j) + E = -(-\tau_i + \lambda' + \tau_j) + E = -\tau_j + \tau_i - \lambda' + E$$

$$= -\tau_j + \tau_i - \tau_i + E = -\tau_j + E.$$

Hence, $-\tau_i + E_i \subseteq -\tau_j + E_j$. Likewise, $-\tau_j + E_j \subseteq -\tau_i + E_i$. Thus, there is some subgroup X of Γ such that $E = \bigcup_{i=1}^{m} (\tau_i + X)$. For future reference we note that, since $|\hat{\nu}(0)| \geq 1$, one of the cosets in the union is X itself.

The next step in our proof is the verification that X is infinite. We suppose on the contrary that X is finite. We shall construct inductively $\{\gamma_0\}$ and $\{\gamma_{k,i}\}_{i=1}^{r} \in \Lambda$ satisfying (3, k), $1 \leq k \leq r^2$. By Lemma 2, we have that $4^{-1} r^{1/2}(1 - e^{-2}) \leq \|\mu\|$, which contradicts our hypothesis, thus proving that X is infinite.

The choice of $\gamma_0 \in \Lambda$ may be arbitrary. Let k be fixed; in general, we choose $\gamma_{k,j} \in \Lambda \backslash (P_{k-1} - E)$ such that

$$\text{(5, k, j)} \qquad |(\bar{\gamma}_{k,j}\mu)\hat{} - \hat{\nu}| < 1 - \varepsilon \quad \text{on} \bigcup_{j<q} (P_{k-1} - \gamma_{k,q}).$$

In choosing $\gamma_{k,j}$, proceed inductively in reverse order (note that (5, k, r) is vacuous). Since X is finite by assumption, so is $P_{k-1} - E$. And, since ν is a weak* cluster point of $\bar{\Lambda}\mu$, the choice of $\{\gamma_{k,j}\}_{j=1}^{r}$ satisfying (5, k, j) is assured.

It remains to check that the γ's chosen above satisfy (3, k). Note that $\gamma_{k,j} \notin P_{k-1} - E$ implies that $(P_{k-1} - \gamma_{k,j}) \cap E = \emptyset$, and hence that $|\hat{\nu}(\gamma)| \leq \varepsilon$ for $\gamma \in P_{k-1} - \gamma_{k,j}$. If $i < j$ and $p \in P_{k-1}$, we have by (5, k, i) that

$$|\hat{\mu}(\gamma_{k,i} + p - \gamma_{k,j})| = |(\bar{\gamma}_{k,i}\mu)\hat{}(p - \gamma_{k,j})|$$

$$\leq |((\bar{\gamma}_{k,i}\mu)\hat{} - \hat{\nu})(p - \gamma_{k,j})| + |\hat{\nu}(p - \gamma_{k,j})|$$

$$< (1 - \varepsilon) + \varepsilon = 1.$$

Since μ satisfies (1, ε), we have that $p + \gamma_{k,i} - \gamma_{k,j} \notin \Lambda$, as desired. This completes the proof that X is infinite.

We next show that, for any finite union of cosets of X, $F = \bigcup (\zeta_j + X)$, $\lambda_n \notin F$ for large n. Recall that $\overline{\lambda_n}\,\mu$ converges weak* to ν. We suppose on the contrary that there is some subsequence of $\{\lambda_n\}$ (still called $\{\lambda_n\}$) such that $\lambda_n \in F$ for all n. Since F is a *finite* union of cosets of X, we may assume, by dropping to a further subsequence if necessary, that $\lambda_n \in \zeta_j + X$ for some j and all n.

Now, since $\overline{\lambda_n}\,\mu$ converges weak* to ν, we see $\overline{\lambda_n}(\overline{-\zeta_j})\mu$ converges weak* to $(\overline{-\zeta_j})\nu$; *i.e.*, $(\overline{\lambda_n - \zeta_j})\mu$ converges weak* to $(\overline{-\zeta_j})\nu$. Hence, $(\overline{\lambda_n - \zeta_j})(\overline{\zeta_j}\,\mu)$ converges weak* to ν. Since $\lambda_n - \zeta_j \in X$, we see that $(\overline{\lambda_n - \zeta_j})\phi(\overline{\zeta_j}\,\mu)$ converges weak* to $\phi(\nu)$, where ϕ is the canonical map from M(G) to $M(G/X^{\perp})$.

Recall that $|\phi(\nu)^{\wedge}|^2$ and $|\phi(\overline{\zeta_j}\,\mu)^{\wedge}|^2$ are weakly almost periodic functions on X, since they are the Fourier-Stieltjes transforms of measures on $G/X^{\perp}$. Therefore, by Theorem 5.3 of [3], there are unique constant functions in the weak closures of the convex hulls of the set of translates (by elements of X) of $|\phi(\nu)^{\wedge}|^2$ and $|\phi(\overline{\zeta_j}\,\mu)^{\wedge}|^2$. Since $|\phi(\nu)^{\wedge}| \geq 1$ on X ($\hat{\nu}|_X = \phi(\nu)^{\wedge}$), this constant function for the former must be at least 1. Since $|\mu|(g + X^{\perp}) = 0$ for all $g \in G$, we have that $\phi(\overline{\zeta_j}\,\mu) \in M_c(G/X^{\perp})$. By Lemma 1, this constant for the latter function is 0.

We shall show that in fact $|\phi(\nu)^{\wedge}|^2$ lies in the weak closure of the convex hull of the set of translates of $|\phi(\overline{\zeta_j}\,\mu)^{\wedge}|^2$. This contradicts the above paragraph. To see this, recall that $(\overline{\lambda_n - \zeta_j})\phi(\overline{\zeta_j}\,\mu)$ converges to $\phi(\nu)$ in the weak* sense as $n \to \infty$. Since the convex hull of the set of translates of $\phi(\overline{\zeta_j}\,\mu)^{\wedge}$ is weakly relatively compact, the sequence $\{(\overline{\lambda_n - \zeta_j})\phi(\overline{\zeta_j}\,\mu)\}^{\wedge}$ has a weak cluster point which belongs to the weak closure of that set. Since the sequence $(\overline{\lambda_n - \zeta_j})\phi(\overline{\zeta_j}\,\mu)$ already converges weak* to $\phi(\nu)$, that cluster point must be $\phi(\nu)^{\wedge}$ itself. Thus, $\phi(\nu)^{\wedge}$ belongs to the weak closure of the convex hull of translates of $\phi(\overline{\zeta_j}\,\mu)^{\wedge}$. Since

$$(\overline{\lambda_n - \zeta_j})\{\phi(\overline{\zeta_j}\,\mu) * \phi(\overline{\zeta_j}\,\mu)^{\sim}\} = \{(\overline{\lambda_n - \zeta_j})\phi(\overline{\zeta_j}\,\mu)\} * \{(\overline{\lambda_n - \zeta_j})\phi(\overline{\zeta_j}\,\mu)^{\sim}\},$$

it follows from the joint weak continuity of multiplication (see Lemma 12.1 of [3]) that $|\phi(\nu)^{\wedge}|^2$ lies in the weak closure of the convex hull of the set of translates of $|\phi(\overline{\zeta_j}\,\mu)^{\wedge}|^2$. This now establishes the claim that $\lambda_n \notin F$, a finite union of cosets of X, for large n.

The proof of the theorem will now be completed by the inductive construction of γ_0 and $\{\gamma_{k,j}\}_{j=1}^{r}$, $1 \leq k \leq r^2$, contained in Λ and satisfying (3, k) for $1 \leq k \leq r^2$. This is, of course, a contradiction. We let $\gamma_0 \in \Lambda$ be arbitrary, and we choose $\{\gamma_{k,j}\}_{j=1}^{r}$, $1 \leq k \leq r^2$, satisfying (5, k, j) as before, with $\gamma_{k,j} \in \Lambda \setminus (P_{k-1} - E)$. This can be done, because, at each stage of the induction, $P_{k-1} - E$ is a finite union of cosets of X, and thus $\lambda_n \notin P_{k-1} - E$ for large n. This completes the proof of the theorem.

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