A NON-NOETHERIAN TWO-DIMENSIONAL HILBERT DOMAIN WITH PRINCIPAL MAXIMAL IDEALS

Robert Gilmer and William Heinzer

All rings considered in this paper are assumed to be commutative and to contain an identity element.

A. V. Geramita (personal communication) has raised the question of whether a Hilbert domain R is Noetherian if each maximal ideal of R is finitely generated. This question arises naturally in at least two contexts. First, the question arises in connection with the well-known theorem of I. S. Cohen to the effect that a ring S is Noetherian if each prime ideal of S is finitely generated [3, Theorem 2]; to wit, O. Goldman introduced the term *Hilbert ring* in [13, p. 136], and his definition of the term was a ring in which each prime ideal is an intersection of maximal ideals. (W. Krull independently considered the class of Hilbert rings in [18]; the terminology of [18, p. 354] for such rings is *Jacobsonsche Ringe*. In different terminology, a Hilbert ring is a ring in which each prime ideal is a J-*radical ideal,* or a J-*prime ideal* [22, p. 631]; for yet another perspective of Hilbert rings, see Section 1-3 of [17].) Second, the property that each of its maximal ideals is finitely generated is inherited by each polynomial ring $R[X_1, \cdots, X_n]$ in finitely many indeterminates over a Hilbert ring R [17, Exercise 8, p. 20]; a straightforward proof of this result can be obtained from the fact that a ring S is a Hilbert ring if and only if $M \cap S$ is a maximal ideal of S for each maximal ideal M of $S[X_1, \cdots, X_n]$ (see [13, Theorem 5] or [18, Section 2]), but an alternate proof would follow at once from the Hilbert Basis Theorem *if* the answer to Geramita's question were affirmative. In Example 1, we construct a Hilbert domain that shows that the answer to Geramita's question is negative. (We use the term *Hilbert domain* to refer to a Hilbert ring that is also an integral domain.) Since a one-dimensional Hilbert domain (or a zero-dimensional Hilbert ring) with finitely generated maximal ideals is Noetherian by Cohen's theorem, such a domain D must have (Krull) dimension at least 2. We show, in fact, that there is a two-dimensional example D_0 that is a Bezout domain (and hence maximal ideals of D_0 are principal) and a subring of Q(X), the rational function field in one variable over the rational field Q. (Examples of one-dimensional, non-Noetherian, Bezout, Hilbert rings with principal maximal ideals are fairly easy to obtain from the well-known $D + M$ construction of [5, Appendix 2]; such rings must contain zero divisors, and a specific example of such a ring is mentioned in the paragraph following Example 1.)

Throughout the remainder of the paper, we use the following notation. Let D be a Dedekind domain with quotient field K, and for each element α in A, an infinite set, let E_α be an infinite family of maximal ideals of D, where $E_\alpha \cap E_\beta = \emptyset$ if α and β are distinct elements of A. Let $\{d_\alpha\}_{\alpha \in A}$ be a subset of D such that $d_\alpha \neq d_\beta$ for $\alpha \neq \beta$, and for each α in A, let $V_\alpha = K[X]_{(X-d_\alpha)}$; thus, V_α is a rank-one discrete valuation ring of the form $K + M_\alpha$, where $M_\alpha = (X - d_\alpha) K[X]_{(X-d_\alpha)}$

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is the maximal ideal of V_α. Finally, let $D_\alpha = \bigcap\{D_M : M \in E_\alpha\}$, let $J_\alpha = D_\alpha + M_\alpha$, and let $J = \bigcap_{\alpha\in A} J_\alpha$. The structure of the domains J_α is well-known (see Theorem A, Appendix 2, of [5]); we use this structure theorem to prove, in results numbered 1 through 10, that J is a two-dimensional Prüfer domain (hence, J is not Noetherian) that is also a Hilbert ring. Then by making some additional assumptions concerning the domain D and the sets E_α, we obtain in Example 1 domains with the properties named in the title of the paper.

RESULT 1. *The domain* D[X] *is a subring of* J, *the quotient field of* J *is* K(X), *and the valuative dimension of* J *is* 2.

Proof. Clearly D is contained in J, and since $X = d_\alpha + [X - d_\alpha]$ is in J_α for each α, it follows that D[X] is a subring of J; whence K(X) is the quotient field of J. Since $\dim_v D[X] = 2 = \dim_v J_\alpha$ for each α, and since $D[X] \subseteq J \subseteq J_\alpha$, it follows that the valuative dimension of J is 2.

RESULT 2. *If* $M \in E_\alpha$, *then* MJ *is a maximal ideal of* J *of height* 2. *The unique height-one prime of* J *contained in* MJ *is* $M_\alpha \cap J$. *For each positive integer* n, *the residue class rings* J/M^nJ *and* D/M^n *are isomorphic.*

Proof. We establish first the last statement of Result 2. Since D is a Dedekind domain, the ideal M^n is invertible. Hence, M^nJ is also invertible, so that $M^nJ = M^nJ\left(\bigcap_\beta J_\beta\right) = \bigcap_\beta (M^nJ)J_\beta = \bigcap_\beta M^nJ_\beta$ [9, Exercise 17, p. 80]. Since M^n is a subset of D_β for each β in A, it follows that

$$M^nJ_\beta = M^n(D_\beta + M_\beta) = M^nD_\beta + M_\beta .$$

Because D is a Dedekind domain and the sets E_β are pairwise disjoint, we have $M^nD_\beta = D_\beta$ for $\beta \neq \alpha$, while $M^nD_\alpha = (MD_\alpha)^n$ [4, Theorem 4]. Returning to the equality $M^nJ = \bigcap_\beta M^nJ_\beta$, we conclude that

$$M^nJ = M^nJ_\alpha \cap \left(\bigcap_{\beta\neq\alpha} J_\beta\right) = M^nJ_\alpha \cap J .$$

Therefore $M^nJ \cap D = (M^nJ_\alpha \cap J) \cap D = (M^nD_\alpha + M_\alpha) \cap D = M^nD_\alpha \cap D$; because D is a Dedekind domain and since $D_M = (D_\alpha)_{MD_\alpha}$ is a rank-one discrete valuation ring with maximal ideal $MD_M = (MD_\alpha)(D_\alpha)_{MD_\alpha}$, it follows that

$$M^nJ \cap D = M^nD_\alpha \cap D = M^nD_M \cap D = M^n .$$

Thus, to within isomorphism,

$$D/M^n \subseteq J/M^nJ \subseteq J_\alpha/M^nJ_\alpha = D_\alpha/M^nD_\alpha = (D_\alpha)_{M_\alpha}/M^n(D_\alpha)_{M_\alpha}$$
$$= D_M/M^nD_M = D/M^n,$$

and D/M^n and J/M^nJ are isomorphic, as asserted. In particular, $J/MJ \simeq D/M$ so that MJ is an invertible maximal ideal of J. Therefore, $\bigcap_{n=1}^\infty M^nJ$ is the unique maximal prime ideal properly contained in MJ [9, Theorem (7.6)]. As noted above, $\bigcap_1^\infty M^nJ = \bigcap_1^\infty (M^nJ_\alpha \cap J) = \left(\bigcap_1^\infty M^nJ_\alpha\right) \cap J = M_\alpha \cap J$; moreover,

$M_\alpha \cap J \neq (0)$ — for example, $X - d_\alpha$ is in $M_\alpha \cap J$. Hence, MJ has height at least 2. Since $\dim J \leq \dim_v J$, which is 2 by Result 1, we conclude that $\dim J = 2$ and that MJ has height 2, as asserted.

Henceforth we use P_α to denote the height-one prime ideal $M_\alpha \cap J$ of J, and we use the letter E to denote $\bigcup_{\alpha \in A} E_\alpha$.

RESULT 3. *No rank-one valuation overring of* J *is centered on an ideal of the form* MJ, *for* M *in* E.

Proof. If some rank-one valuation overring V of J were centered on MJ, where M is in E_α, then the equality $\bigcap_{n=1}^\infty M^n V = (0)$ would imply that $P_\alpha = \bigcap_{n=1}^\infty M^n J = (0)$, contrary to Result 2.

RESULT 4. *The equality* $J_{P_\alpha} = V_\alpha$ *holds for each* α *in* A.

Proof. We clearly have $J_{P_\alpha} \subseteq (J_\alpha)_{M_\alpha} = V_\alpha$. Moreover, since $M_\alpha \cap K = (0)$, it follows that $P_\alpha \cap D = (0)$ so that $K[X] \subseteq J_{P_\alpha}$, a one-dimensional quasilocal overring of K[X]. Thus, J_{P_α} is a rank-one discrete valuation ring contained in V_α, and $J_{P_\alpha} = V_\alpha$, as we wished to prove.

The proof of the next result uses the following lemma.

LEMMA 1. *Assume that* R *is a quasilocal domain with principal maximal ideal* $mR \neq (0)$ *and that* $Q = \bigcap_{n=1}^\infty m^n R$. *Then* $QR_Q \subseteq R$, *and if* R_Q *is a valuation ring, then so is* R.

Proof. Consider $q \in Q$, $s \in R - Q$. If $s \notin mR$, then s is a unit of R and $q/s \in R$. If $s \in mR$, then $s \notin Q = \bigcap_1^\infty m^n R$ implies that there exists a positive integer k such that $s \in (m^k R - m^{k+1} R)$. Thus $s = m^k t$, where t is a unit of R, and $q/s = (q/m^k)t^{-1}$, where $q/m^k \in R$ since $q \in Q \subseteq m^k R$; consequently, $q/s \in R$ and $QR_Q \subseteq R$, as asserted. Note, in fact, that $QR_Q = Q$ since Q is prime in R. Thus if R_Q is a valuation ring, then R is a valuation ring, for R is the inverse image, under the canonical homomorphism, of the rank-one discrete valuation ring R/Q on the field $R_Q/QR_Q = R_Q/Q$ [21, (11.4)].

RESULT 5. *For each* M *in the set* E, *the ring* J_{MJ} *is a valuation ring.*

Proof. Result 5 follows at once from Results 2 and 4 and from Lemma 1.

RESULT 6. *If* V *is a nontrivial valuation overring of* J *that is not in the set* $\{V_\alpha\}_{\alpha \in A} \cup \{J_{MJ}: M \in E\}$, *then* V *is of the form* $K[X]_{(f(X))}$, *where* f(X) *is an irreducible monic polynomial in* K[X] *distinct from each* $X - d_\alpha$. *Such a valuation ring is a quotient ring of* J, *and hence* J *is a Prüfer domain.*

Proof. Let P be the maximal ideal of V and let $D^* = \bigcap_{\alpha \in A} D_\alpha$. Then $P \cap D^*$ is prime in D^* so that $P \cap D^* = (0)$ or $P \cap D^* = MD^*$ for some $M \in E$. In the second case $P \cap J$ contains MJ, so that $V \supseteq J_{MJ}$, a rank-two valuation ring. Therefore, $V = J_{MJ}$ or $V = V_\alpha$, where $M \in E_\alpha$. It follows that the first case occurs; that is, $P \cap D^* = (0)$. Thus $K[X] \subseteq V$, so that $V = K[X]_{(f(X))}$ for some irreducible monic polynomial $f(X) \in K[X]$. Since $V \neq V_\alpha = K[X]_{(X - d_\alpha)}$ for each $\alpha \in A$ by assumption, it follows that f(X) is not one of the polynomials $X - d_\alpha$.

We have $D^* - \{0\} \subseteq J - P$, and hence $K[X] \subseteq J_{P\cap J} \subseteq V$. Since V is a quotient ring of K[X], it is also a quotient ring of $J_{P\cap J}$, and hence of J. From Results 4 and 5, it then follows that each valuation overring of J is a quotient ring of J; whence J is a Prüfer domain [5, p. 334].

According to the terminology of [10], a ring R has the *n-generator property* if each finitely generated ideal of R can be generated by n elements. An outstanding question in the theory of Prüfer domains is whether a Prüfer domain has the 2-generator property (see, for example, [6], [12], [7], [8], [19], [1], [24], [16], [15], [2], and [25]). Thus, each time a new construction of Prüfer domains appears in the literature, it is natural to ask if the construction yields Prüfer domains without the 2-generator property. We therefore interrupt our main line of development to prove that the Prüfer domains J of this paper have the 2-generator property.

RESULT 7. *If* B *is a nonzero ideal of* D, *then* J/BJ *is a homomorphic image of* D/B *so that* J/BJ *is a principal ideal ring. Each finitely generated fractional ideal of* J *can be generated by two elements. If* J *is a Bezout domain, then each* D_α *is a principal ideal domain; if* $D^* = \bigcap \{D_M: M \in E\}$ *is a principal ideal domain, then* J *is a Bezout domain.*

Proof. The ideal B is uniquely expressible in the form $B_1 B_2$, where each maximal ideal of D containing B_1 is in E, while no maximal ideal of D containing B_2 is in E. An argument similar to that used in the proof of Result 2 shows that $B_2 J = J$, and hence $BJ = B_1 J$. Since $D/B_1 \simeq (D/B)/(B_1/B)$ is a homomorphic image of D/B, the first statement of Result 7 will follow from the relation $D/B_1 \simeq J/B_1 J$, which we proceed to establish. Thus B_1 is a finite product $M_1^{e_1} \cdots M_t^{e_t}$ of distinct maximal ideals M_i in E; moreover, the ideals $M_i^{e_i}$, $1 \leq i \leq t$, are pairwise comaximal so that $D/B_1 \simeq (D/M_1^{e_1}) \oplus \cdots \oplus (D/M_t^{e_t})$. Similarly,

$$J/B_1 J \simeq (J/M_1^{e_1} J) \oplus \cdots \oplus (J/M_t^{e_t} J),$$

and since $D/M_i^{e_i} \simeq J/M_i^{e_i} J$ by Result 2, the relation $D/B_1 \simeq J/B_1 J$ then follows.

Since K(X) is the quotient field of J, to prove that each finitely generated fractional ideal of J is generated by two elements, it suffices to prove that if $\{f_1, \cdots, f_t\}$ is a finite set of nonzero elements of K[X], where $t \geq 2$, then $F = \{f_1, \cdots, f_t\}J$ can be generated by two elements. Moreover, there is no loss of generality in assuming that the greatest common divisor of $f_1, \cdots, f_t$ in K[X] is 1. Thus, $1 = \sum_{i=1}^{t} f_i g_i$ for some $g_1, \cdots, g_t$ in K[X]. Choose nonzero elements d_1, d_2 of D such that $d_1 f_i$ and $d_2 g_i$ are in D[X] for each i. Then $d_1 d_2 = \sum_1^t (d_1 f_i)(d_2 g_i)$ is in $d_1 F$, an integral ideal of J. Since $J/d_1 d_2 J$ is a principal ideal ring, it follows that $d_1 F$ can be generated by two elements (one of which can be chosen to be $d_1 d_2$), and consequently, F also can be generated by two elements.

If J is a Bezout domain, then each overring of J is also a Bezout domain; in particular, each $D_\alpha + M_\alpha$ is a Bezout domain, and this implies that each D_α is a (Noetherian) Bezout domain [12, p. 148]; that is, a principal ideal domain (PID).

In proving that D^* a PID implies that J is a Bezout domain, note that there is no loss of generality in assuming that $D = D^*$. This is true since $\{MD^*: M \in E\}$ is the set of maximal ideals of D^* and since

$$D_\alpha = \bigcap \{D_M: M \in E_\alpha\} = \bigcap \{(D^*)_{MD^*}: M \in E_\alpha\} \quad \text{for each } \alpha .$$

Thus, we assume that D is a PID and that E is the set of maximal ideals of D. To prove that J is a Bezout domain, it suffices to prove that if f and g are nonzero elements in D[X] with greatest common divisor 1, then $B = \{f, g\} J$ is principal. As shown in the second paragraph of this proof, the ideal B contains a nonzero element b of D. Then $bD = M_1^{e_1} \cdots M_t^{e_t}$ is a finite product of maximal ideals of D and $bJ = (M_1 J)^{e_1} \cdots (M_t J)^{e_t} \subseteq B$. It then follows easily from the fact that J/bJ is a principal ideal ring with maximal ideals $M_1 J/bJ, \cdots, M_t J/bJ$ that there exist integers $f_1, \cdots, f_t$, with $0 \le f_i \le e_i$ for each i, such that

$$B = (M_1 J)^{f_1} \cdots (M_t J)^{f_t} = M_1^{f_1} \cdots M_t^{f_t} J .$$

Therefore, B is principal since each M_i is principal. This completes the proof of Result 7.

The reduction in Result 7 to the case where D is equal to D^* could have been made initially; thus we could have assumed, without loss of generality, that $\{E_\alpha\}_{\alpha \in A}$ is a partition of the set of *all* maximal ideals of D, but the (apparently) more general approach seemed advantageous to us. We remark that J may be a Bezout domain although D^* is not a PID; for example, if D^* has finite class group, then D^* can be expressed as the intersection of two overrings D_1 and D_2 that are principal ideal domains [9, Exercise 5, p. 505], and Theorem 3.6 of [8] implies that J is a Bezout domain for this choice of D^*, D_1, and D_2.

We have considered variations on the construction of J in attempting to provide an example of a Prüfer domain that does not have the 2-generator property. One such variation is to replace the Dedekind domain D by a Prüfer domain D' of dimension greater than 1 such that maximal ideals of D' are finitely generated; a difficulty in this approach is in showing that the resulting domain (call it J') is a Prüfer domain. The question of whether each Prüfer domain that is a subring of Q(X), say, has the 2-generator property may be regarded as a test case for certain modifications in the construction; that is, any modification in the construction of J that yielded an answer to the preceding question would be a significant step.

We return to the main theme of the paper, Geramita's question. The initial assumptions that the set A and each of the sets E_α is infinite are used for the first time in the proof of the next result. (For A finite, however, our construction is a special case of Section 3 of [8].)

RESULT 8. *The ring* J *is a Hilbert domain.*

Proof. It suffices to prove: (a) each P_α is an intersection of maximal ideals of J, and (b) $\bigcap_{\alpha \in A} P_\alpha = (0)$. Statement (a) follows since $M_\alpha = \bigcap \{MJ_\alpha: M \in E_\alpha\}$, so that $P_\alpha = M_\alpha \cap J = \bigcap_{M \in E_\alpha} (MJ_\alpha \cap J) = \bigcap_{M \in E_\alpha} MJ$, where each MJ is maximal in J. (The equality $M_\alpha = \bigcap_{M \in E_\alpha} MJ_\alpha$ depends upon the assumption that E_α is infinite.) Since $\{V_\alpha\}_{\alpha \in A}$ is an infinite subset of the family of nontrivial valuation overrings of the domain K[X], it follows easily that

$$(0) = \bigcap_{\alpha \in A} M_\alpha = \bigcap_\alpha (M_\alpha \cap J) = \bigcap_\alpha P_\alpha ,$$

and this is the content of (b).

Results 1 through 8 show that the domain J is a two-dimensional non-Noetherian Hilbert ring that is also a Prüfer domain with the 2-generator property. Moreover, $\{MJ: M \in E\}$ is a family of finitely generated maximal ideals of J, and Result 6 shows that the other maximal ideals (if any) arise as the centers on J of the essential valuation rings $K[X]_{(f(X))}$ of $K[X]$ (where f(X) is monic irreducible and distinct from each $X - d_\alpha$) that contain J; we point out later that (infinitely many) such maximal ideals may exist. The next two results will be used in Example 1 to show that suitable restrictions placed on the sets E_α imply that no such valuation ring contains J, and hence $\{MJ: M \in E\}$ is the set of maximal ideals of J.

RESULT 9. *Assume that* $f(X) \in K[X]$. *Then* $f(X) \in J$ *if and only if* $f(d_\alpha) \in D_\alpha$ *for each* α. *If* $f(X) \in J$, *then* $f(X)$ *is a unit of* J *if and only if* $f(d_\alpha)$ *is a unit of* D_α *for each* α.

Proof. It is clear that $f(X) \in J \Leftrightarrow f(X) \in J_\alpha$ for each α, and that f(X) is a unit of $J \Leftrightarrow f(X)$ is a unit of each J_α.

Now $f(X) = f(d_\alpha) + (X - d_\alpha)[(f(X) - f(d_\alpha))/(X - d_\alpha)]$, where

$$[(f(X) - f(d_\alpha))/(X - d_\alpha)]$$

is in K[X]. Therefore,

$$f(X) \in J_\alpha \Leftrightarrow f(d_\alpha) \in J_\alpha \Leftrightarrow f(d_\alpha) \in J_\alpha \cap K = D_\alpha .$$

If $f(X) \in J_\alpha$, then f(X) is a unit of $J_\alpha \Leftrightarrow f(d_\alpha)$ is a unit of D_α.

RESULT 10. *Assume that* $f(X) \in K[X] - K$, *that* $f(d_\alpha) \neq 0$ *for each* α *in* A, *and that there are only finitely many elements* α *in* A *such that* $f(d_\alpha)$ *is a nonunit of* D_α; *if this finite set is* $\{\alpha_i\}_{i=1}^t$, *then* $\xi = (X - d_{\alpha_1}) \cdots (X - d_{\alpha_t})/f(X)$ *is in* J.

Proof. For each α in $A - \{\alpha_i\}_{i=1}^t$, the element $f(d_\alpha)$ is a unit of D_α, so by Result 9, the element ξ is in J_α for each such α. And since $f(d_{\alpha_i}) \neq 0$, f(X) is a unit of V_{α_i} so that $\xi \in M_{\alpha_i} \subset J_{\alpha_i}$. Consequently, $\xi \in \bigcap_{\alpha \in A} J_\alpha = J$, as asserted.

While it is possible to continue a structure theory of J in the general setting of D, $\{E_\alpha\}_{\alpha \in A}$, and $\{d_\alpha\}_{\alpha \in A}$, we choose to pass now to a more concrete setting; while D, A, and the elements d_α are specified in Example 1, much flexibility remains in the choice of the sets E_α.

Example 1. In the notation used up to this point, take D to be Z, the ring of integers, let A be Z^+, the set of positive integers, and let $d_i = i$ for each i in Z^+. We shall specify the elements of $E_1, E_2, \cdots$ by means of their positive generators; that is, each E_i will be described as an infinite set of positive prime integers, and for $i \neq j$, the sets E_i and E_j are to be disjoint. We use the symbol Z_t instead of D_t, but otherwise our notation—J_t, V_t, M_t, P_t, K, *etc.*—is consistent with the notation already established. Our guiding principle in the choice of the sets $E_1, E_2, \cdots$ is to insure that for each polynomial f(X) in Q[X] that has no positive integer as a root, f(t) is a nonunit of Z_t for only finitely many integers t. We prove at once that if the sets E_i are so chosen, then no essential valuation ring $Q[X]_{(g(X))}$ of Q[X], where g(X) is monic, irreducible, and $g(i) \neq 0$ for each i in Z^+, contains J, and hence the (Bezout) domain J provides an example showing that the answer to Geramita's question is negative. Thus, if g(t) is a unit of Z_t for each positive integer t, then Result 9 implies that $1/g(X)$ is in J, but not in $Q[X]_{(g(X))}$. Otherwise, let $\{t_i\}_{i=1}^{k}$ be the finite set of positive integers t such that g(t) is a nonunit of Z_t; Result 10 implies that the element $\xi = (X - t_1) \cdots (X - t_k)/g(X)$ is in J, but ξ is not in $Q[X]_{(g(X))}$. We therefore proceed to establish the existence of sets $E_1, \cdots, E_n, \cdots$ satisfying the required conditions.

Let $\{f_i(X)\}_{i=1}^{\infty}$ be the set of monic irreducible polynomials in Q[X] that are distinct from X - 1, X - 2, ⋯, let Π denote the set of prime integers, and for $p \in \Pi$, denote by v_p the p-adic valuation on Q.

Consider $f_1(1)$. The set S_1 of prime integers p such that $v_p(f_1(1)) \neq 0$ is finite. Partition the set $\Pi - S_1$ into two infinite sets E_1 and T_1. Consider $f_1(2)$ and $f_2(2)$. The set S_2 of primes p in T_1 such that v_p has nonzero value on one of these two rational numbers is finite. Partition $T_1 - S_2$ into two infinite sets E_2 and T_2. Then consider $f_1(3)$, $f_2(3)$, and $f_3(3)$. The set S_3 of primes p in T_2 such that v_p has nonzero value on one of these rational numbers is finite. Partition $T_2 - S_3$ into infinite subsets E_3 and T_3, *etc.* We claim that for $n, t \in Z^+$, with $n \leq t$, $f_n(t)$ is a unit of Z_t. If $p \in E_t$, then $p \in T_{t-1} - S_t$, and hence $v_p(f_n(t)) = 0$. Therefore, $f_n(t)$ is a unit of Z_t since $f_n(t)$ is a unit of Z_{pZ} and $Z_t = \bigcap_{p \in E_t} Z_{pZ}$. Since each $f_i(X)$ has the property that $f_i(t)$ is a nonunit of Z_t for only finitely many integers t, it follows that each nonconstant polynomial $f(X) \in Q[X]$ with no root in Z^+ has the same property. Thus endeth the construction of the desired example.

As noted in the introduction, Cohen's theorem implies that a one-dimensional Hilbert domain is Noetherian if each of its maximal ideals is finitely generated. On the other hand, examples of one-dimensional non-Noetherian Hilbert rings R in which each maximal ideal is principal are fairly easy to come by. For instance, let $D = Z + XQ[[X]]$ and let $R_0 = D/X^2 D$; that R_0 is an appropriate example follows from the well-known structure of the domain D. We remark that other such examples are easily obtained from the domains J of Example 1; for since J is a two-dimensional Bezout domain in which each maximal ideal has height 2 (in alternate terminology [23, p. 510], [9, p. 383], the domain J *satisfies the first chain condition for prime ideals*), each height-one prime P of J is not finitely generated [9, p. 289], so that if y is a nonzero element of P, then J/yJ is an example of the kind of ring under discussion. The rings J/yJ are, of course, more complex than the ring R_0 because of the theory needed to develop the structure of the domain J.

Do there exist examples of arbitrary dimension $k \geq 2$ showing that the answer to Geramita's question is negative? Yes, $J[X_1, \cdots, X_{k-2}] = J^{(k-2)}$, for J as in Example 1, is a k-dimensional non-Noetherian Hilbert domain, and each maximal ideal of $J^{(k-2)}$ has a basis of k - 1 elements. The domain $J^{(k-2)}$ is not a Prüfer domain (if $k > 2$); however, as an alternative to the explicit ring theoretic construction we give here, another way to obtain examples answering the Geramita question is to make use of the construction given in [14]. This construction via partially ordered abelian groups can be used to yield k-dimensional Hilbert, Bezout domains in which all maximal ideals are principal for each positive integer k. (The construction in [14] uses what Mott in [20] calls the Krull-Kaplansky-Jaffard-Ohm Pullback Theorem; a key point in obtaining the k-dimensional Bezout domain in question is to start the construction in [14] with a group that is an infinite direct sum of copies of Z, ordered lexicographically.)

Two questions that naturally arise in connection with Result 6 and Example 1 are the following.

(Q1) *In the notation of Result* 6, *is it, in fact, possible for* J *to have valuation overrings of the form* $K[X]_{(f(X))}$?

(Q2) *If the answer to* (Q1) *is affirmative, is it nevertheless possible that the center on* J *of each such valuation overring* $K[X]_{(f(X))}$ *of* J *is finitely generated, so that* J *already provides a negative answer to Geramita's question, with no additional restrictions imposed on* D *or on the sets* E_α ?

In view of our further development of the paper after Result 6, the expected answer to (Q1) is affirmative, and we substantiate this expectation in Example 2 below by proving that even for D = Z, the domain J may have infinitely many valuation overrings of the form $K[X]_{(f(X))}$. While one might predict (correctly, as it turns out) on the same basis that (Q2) has a negative answer, the actual situation in regard to (Q2) is perhaps surprising:

RESULT 11. *If a valuation ring* $V = K[X]_{(f(X))}$ *contains* J, *with* V *not in the set* $\{V_\alpha\}_{\alpha\in A}$, *then the center of* V *on* J *is not finitely generated.*

Proof. Let P be the center of V on J. Since $V = J_P$ contains

$$J = \bigcap_{\alpha\in A} J_\alpha = \bigcap_{\alpha\in A} \left[\bigcap \{(J_\alpha)_{MJ_\alpha} : M \in E_\alpha\} \right]$$

$$= \bigcap_{\alpha\in A} \left[\bigcap \{J_{MJ} : M \in E_\alpha\} \right] = \bigcap \{J_{MJ} : M \in E\},$$

and since J is a Prüfer domain, it follows from Proposition 1.4 of [11] that each finitely generated ideal contained in P is contained in MJ for some M in E; since P is maximal in J and is distinct from each MJ, it follows that P is not finitely generated.

We conclude the paper with a concrete example that proves the answer to (Q1) is affirmative.

Example 2. As in Example 1, we take D to be Z, A to be Z^+, and we let $d_i = i$ for each i in Z^+; similar conventions with respect to the symbols $\{E_1, E_2, \cdots\}$, Z_t and D_t, J_t, V_t, *etc.* also apply in this example. But for Example 2, we first observe that if $E_1, E_2, \cdots$ are infinite, pairwise disjoint subsets of the set Π of positive prime integers satisfying the following condition (η), then $Q[X]_{(X+k)}$ contains the domain J for each $k \geq 0$.

(η) *For each* $k \geq 0$, *there exist infinitely many positive integers* r *such that* r + k *has a prime factor in* E_r.

Having made this observation, we then proceed to establish existence of such sets $E_1, E_2, \cdots$.

Thus, assume that $E_1, E_2, \cdots$ are infinite, pairwise disjoint subsets of Π satisfying condition (η). We choose a nonzero element α in

$$J = \bigcap_{i=1}^{\infty} \left[\bigcap \{Z_{pZ} + (X - i)Q[X]_{(X-i)} : p \in E_i\} \right],$$

we choose $k \geq 0$, and we prove that α is in $Q[X]_{(X+k)}$. Thus, we express α in the form $(f_1/g_1)(X+k)^m$, where f_1 and g_1 are elements of Z[X] that do not vanish at

$\div k$, and $m \in Z$; our object is to prove that $m \geq 0$. Since the family $\{E_i\}_{i=1}^{\infty}$ satisfies condition (η), the set B of integers r satisfying the following four conditions is infinite: $f_1(r) \neq 0$, $g_1(r) \neq 0$, $r + k$ has a prime divisor in E_r, and $f_1(-k)$ has no prime divisor in E_r.

We write $f_1(X)$ as $a_0 + a_1(X + k) + \cdots + a_u(X + k)^u$, where each a_i is an integer. Assume that $r \in B$ and that p is a prime divisor of $r + k$ in E_r. Result 9 implies that $[f_1(r)/g_1(r)](r + k)^m$ has nonnegative value in the p-adic valuation v_p of Q. If m were negative, it would then follow that

$$f_1(r) = a_0 + a_1(r + k) + \cdots + a_u(r + k)^u$$

has positive v_p-value; since p divides $r + k$, this assertion would imply that p divides $a_0 = f_1(-k)$, contrary to the choice of the set B. Consequently, $m \geq 0$ and $\alpha \in Q[X]_{(X+k)}$, as we wished to prove.

We establish the existence of subsets $E_1, E_2, \cdots$ of the set Π of prime integers such that $\{E_i\}_{i=1}^{\infty}$ satisfies condition (η). We describe each of the sets E_i as a union of subsets $\{S_{ji}\}_{j=0}^{\infty}$, as follows. We partition Π into two infinite subsets A_0 and B_0, and then we partition A_0 into a countably infinite family $\{S_{0i}\}_{i=1}^{\infty}$ of infinite subsets. The sets $\{S_{1i}\}$ are then determined as follows. We partition B_0 into infinite subsets A_1 and B_1; then S_{1i} is $\{i\}$ or $\emptyset$, according as i is, or is not, in A_1. Now partition B_1 into infinite subsets A_2 and B_2, and for $i \in Z^+$, define S_{2i} to be $\{i + 1\}$ if $i + 1$ is in A_2, and $\emptyset$ otherwise. We continue this procedure by induction, obtaining subsets $E_1, E_2, \cdots$ of Π defined by $E_i = \bigcup_{j=0}^{\infty} S_{ji}$ for each i. The family $\{E_i\}_{i=1}^{\infty}$ satisfies condition (η). The proof that $E_i \cap E_j = \emptyset$ for $i \neq j$ is straightforward, and E_i is infinite since S_{0i} is an infinite subset of E_i. Finally, if $k \geq 0$, then for each integer $r > k$ in the infinite set A_{r+1}, the integer $r - k$ is such that $k + (r - k) = r$ has a prime divisor r in $S_{k+1, r-k} \subseteq E_{r-k}$. This completes the presentation of Example 2.

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Robert Gilmer:
Department of Mathematics
Florida State University
Tallahassee, Florida 32306

William Heinzer:
Department of Mathematics
Purdue University
West Lafayette, Indiana 47907