EXTRINSIC SPHERES IN KÄHLER MANIFOLDS

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1. INTRODUCTION

An n-dimensional submanifold M^n of an arbitrary Riemannian manifold $\tilde{M}^m$ is called an *extrinsic sphere* if it is umbilical and has parallel mean curvature vector $H \neq 0$ [4]. (Dimensions of manifolds are real dimensions.) We say that a Riemannian manifold $\tilde{M}^m$ is *sufficiently curved* if for every point $x \in \tilde{M}^m$, the maximal linear subspace V of the tangent space $T_x(\tilde{M}^m)$ of $\tilde{M}^m$ at x with $\tilde{R}(X, Y) = 0$ for $X, Y \in V$ has dimension less than $m - 2$, where $\tilde{R}$ denotes the curvature tensor of $\tilde{M}^m$.

In this paper, we shall study extrinsic spheres in an arbitrary Kähler manifold. In particular, we shall prove the following.

THEOREM 2. *There exists no complete orientable extrinsic sphere of codimension two in any sufficiently curved Kähler manifold.*

Remark 1. A standard $(m - 1)$-sphere $(m \geq 3)$ of small radius can be imbedded as an extrinsic sphere in the complex projective space $P^{2m}(\mathbb{C})$, which is positively curved by the Fubini-Study metric [3]. For the classification of umbilical submanifolds in complex space forms, see [3]. For the nonexistence of extrinsic spheres of codimension two in irreducible Hermitian symmetric spaces of dimension greater than 2, see [2].

2. PRELIMINARIES

Let M^n be an n-dimensional submanifold of a 2m-dimensional Kähler manifold $\tilde{M}^{2m}$ with complex structure J and Kähler metric g, and let ∇ and $\tilde{\nabla}$ be the covariant differentiations on M^n and $\tilde{M}^{2m}$, respectively. Then the second fundamental form σ is defined by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X and Y are vector fields tangent to M^n and σ is a normal-bundle-valued symmetric 2-form on M^n. For a vector field ξ normal to M^n, we write

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential component (respectively, the normal component) of $\tilde{\nabla}_X \xi$. A normal vector field ξ is said to be *parallel* if $D\xi = 0$. The submanifold is said to be *umbilical* if $\sigma(X, Y) = g(X, Y) H$, where $H = (\text{trace } \sigma)/n$ is the *mean curvature vector* of M^n in $\tilde{M}^{2m}$.

Let R, $\tilde{R}$, and R^N be the curvature tensors associated with ∇, $\tilde{\nabla}$, and D, respectively. For example, $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

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For the second fundamental form σ, we define the covariant derivative, denoted by $\overline{\nabla}_X \sigma$, to be

$$(1) \qquad (\overline{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

Then, for all vector fields X, Y, Z, W tangent to M^n and for all vector fields ξ, η normal to M^n, the equations of Gauss, Codazzi, and Ricci take the forms

$$(2) \qquad \tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(X, W), \sigma(Y, Z)),$$

$$(3) \qquad (\tilde{R}(X, Y)Z)^{\perp} = (\overline{\nabla}_X \sigma)(Y, Z) - (\overline{\nabla}_Y \sigma)(X, Z),$$

$$(4) \qquad \tilde{R}(X, Y; \xi, \eta) = R^N(X, Y; \xi, \eta) - g([A_\xi, A_\eta]X, Y),$$

where $\tilde{R}(X, Y; Z, W) = g(\tilde{R}(X, Y)Z, W)$ and $R^N(X, Y; \xi, \eta) = g(\tilde{R}^N(X, Y)\xi, \eta)$, and where $^{\perp}$ in (3) denotes the normal component.

3. EXTRINSIC SPHERES IN KÄHLER MANIFOLDS

THEOREM 1. *Let* M^{2n} *be a* 2n-*dimensional complete extrinsic sphere in any Kähler manifold* $\tilde{M}^{2m}$. *If there exist* 2m - 2n *mutually orthogonal parallel unit normal vector fields along* M^{2n}, *then* M^{2n} *is isometric to the standard* 2n-*sphere of radius* $1/\alpha$, *and* $\tilde{R}(X, Y) = 0$ *for all vectors* X, Y *tangent to* M^{2n}.

Proof. Let M^{2n} be an extrinsic sphere in a Kähler manifold $\tilde{M}^{2m}$. Then by the definition,

$$(5) \qquad \sigma(X, Y) = g(X, Y)H, \quad D_X H = 0, \quad \text{and} \quad H \neq 0,$$

for all vectors X, Y tangent to M^{2n}. Since H is parallel, the length, say α, of H is a nonzero constant. If we put

$$(6) \qquad H = \alpha\bar{\xi},$$

then $\bar{\xi}$ is a parallel unit normal vector field along M^{2n}. Now, suppose that $\xi_1, \cdots, \xi_{2m-2n}$ are 2m - 2n mutually orthogonal parallel unit normal vector fields defined on the whole M^{2n}. Then, by Proposition 1.3 of [1, p. 101], we may assume that $\xi_1 = \bar{\xi}$. We define 2m - 2n - 1 functions on M^{2n} by $\phi_r = g(J\bar{\xi}, \xi_r)$, r = 2, $\cdots$, 2m - 2n. From (5) and the parallelism of ξ_r, we find

$$(7) \qquad \tilde{\nabla}_X \xi_r = -A_{\xi_r} X + D_X \xi_r = 0.$$

Thus, we find

$$(8) \qquad X\phi_r = g(J\tilde{\nabla}_X \bar{\xi}, \xi_r) = \alpha g(X, J\xi_r).$$

From (5), (6), (7), and (8), we have

$$\begin{aligned} XY\phi_r &= \alpha X g(Y, J\xi_r) = \alpha g(\tilde{\nabla}_X Y, J\xi_r) = \alpha g(\nabla_X Y, J\xi_r) + \alpha g(\sigma(X, Y), J\xi_r) \\ &= (\nabla_X Y)\phi_r - \alpha^2 \phi_r g(X, Y), \end{aligned}$$

from which we get

$$\nabla_X d\phi_r = -\alpha^2 \phi_r X, \qquad r = 2, \cdots, 2m - 2n. \tag{9}$$

Now, we shall claim that at least one of the functions ϕ_r, $r = 2, \cdots, 2m - 2n$, is a nonconstant function. If all of the ϕ_r are constant, then (5) and (7) imply

$$\begin{aligned} 0 = X\phi_r &= g(J\tilde{\nabla}_X \bar{\xi}, \xi_r) = -g(JA_{\bar{\xi}} X, \xi_r) \\ &= -\alpha g(JX, \xi_r) = \alpha g(X, J\xi_r), \qquad r = 2, \cdots, 2m - 2n. \end{aligned}$$

Thus, the subspace spanned by $\xi_2, \cdots, \xi_{2m-2n}, J\xi_2, \cdots, J\xi_{2m-2n}$ is a J-invariant normal subspace. Thus, it is even-dimensional and of dimension greater than $2m - 2n - 1$. Hence, it is the whole normal space of M^{2n} in $\tilde{M}^{2m}$. This implies that M^{2n} is a complex submanifold of the Kähler manifold $\tilde{M}^{2m}$, from which we see that M^{2n} is minimal. This is a contradiction. Thus, we know that there exists a nonconstant function ϕ defined on M^{2n} and satisfying the differential equation $\nabla_X d\phi = -\alpha^2 \phi X$ for all vectors X tangent to M^{2n}. Therefore, by a result of Obata [5], M^{2n} is isometric to the standard 2n-sphere of radius $1/\alpha$. Thus, in particular, the curvature tensor R of M^{2n} is given by

$$R(X, Y; Z, W) = \alpha^2 [g(X, W) g(Y, Z) - g(X, Z) g(Y, W)]. \tag{10}$$

Substituting (5) and (10) into the equation of Gauss, we get

$$\tilde{R}(X, Y; Z, W) = 0. \tag{11}$$

From (1) and (5), we find $(\bar{\nabla}_X \sigma)(Y, Z) = g(Y, Z) D_X H = 0$. Thus, the equation of Codazzi implies

$$(\tilde{R}(X, Y) Z)^{\perp} = 0. \tag{12}$$

Since M^{2n} is umbilical, the second fundamental tensors commute. Thus,

$$[A_\xi, A_\eta] = 0, \tag{13}$$

for all vectors ξ, η normal to M^{2n}. On the other hand, the existence of $2m - 2n$ parallel unit normal vector fields $\xi_1, \cdots, \xi_{2m-2n}$ implies that the normal curvature tensor R^N is trivial. Thus, by (13) and the equation of Ricci,

$$\tilde{R}(X, Y; \xi, \eta) = 0. \tag{14}$$

By using the identity $\tilde{R}(X, Y; Z, \xi) + \tilde{R}(X, Y; \xi, Z) = 0$, we see that (11), (12), and (14) imply

$$\tilde{R}(X, Y) = 0 \tag{15}$$

for all vectors X, Y tangent to M^{2n}. This completes the proof of the theorem.

Remark 2. If M^{2n} is simply connected, the existence of $2m - 2n$ mutually orthogonal unit parallel normal vector fields (defined on the whole manifold M^{2n}) is equivalent to the triviality of R^N (see [1, p. 99 and p. 143]).

Remark 3. The standard 2n-sphere in $\mathbb{C}^{2m}$ satisfies the assumption of Theorem 1.

Now we shall give a proof of Theorem 2.

If M^{2n} is a complete orientable extrinsic sphere of codimension two in any Kähler manifold, then, by the parallelism of the mean curvature vector H and the nonvanishing of H, we see that the normal connection is trivial; *i.e.*, $R^N = 0$. Thus, if we choose ξ_2 to be one of the two unit normal vector fields perpendicular to H, then the assumptions of Theorem 1 are all satisfied. Thus, the ambient space $\tilde{M}^{2n+2}$ is not sufficiently curved. This completes the proof of Theorem 2.

As an immediate consequence of Theorem 2, we have the following.

COROLLARY. *There exist no complete orientable extrinsic spheres of codimension two in any positively (or negatively) curved Kähler manifold.*

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