ON THE BREAKDOWN PHENOMENA OF SOLUTIONS OF QUASILINEAR WAVE EQUATIONS

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0. INTRODUCTION

We consider the quasilinear wave equation $y_{tt} - Q^2(y_x)y_{xx} = 0$ subject to the initial and boundary conditions: $y_x(x, 0) = g(x)$, $y_t(x, 0) = f(x)$, $0 \leq x \leq L$; $y(0, t) = y(L, t) = 0$, $t \geq 0$. We can transform this system to the initial-value problem of a hyperbolic conservation law if $f(x)$ and $g(x)$ satisfy some compatibility conditions.

We consider two cases:

Case 1. $g(x) \equiv 0$ and $f(x)$ satisfies some convexity conditions;

Case 2. $f(x) \equiv 0$ and $g(x)$ satisfies some convexity conditions.

We prove that a necessary condition for the existence of a C^2 global solution is that the solution be periodic in t in some sense, which is the classical one for the linear problem. We present a necessary and sufficient set of conditions for the solution to break down in the sense that some second-order derivatives of the solution become unbounded at a finite time. By this set we mean that if the solution breaks down, then one condition in this set holds. Conversely, if one condition in this set holds, then the solution eventually breaks down. We derive some conditions on Q', which are weaker than those considered in [5], [6], and [4], which are sufficient for the solution to break down.

F. John [2] has obtained the breakdown result for the general, genuinely nonlinear conservation laws with n-characteristics and with initial functions which are sufficiently small. For our special conservation law, the genuine nonlinearity condition is $Q' \neq 0$. We derive the breakdown results under some conditions on Q' weaker than $Q' \neq 0$.

1. DEFINITIONS AND NOTATION

We define $u(x, t) = y_x(x, t)$ and $v(x, t) = y_t(x, t)$. The problem is equivalent to the system

$$(1) \qquad \begin{aligned} u_t - v_x &= 0, \\ v_t - Q^2(u)u_x &= 0; \end{aligned}$$

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$$u(x, 0) = g(x), \quad v(x, 0) = f(x), \quad 0 \leq x \leq L,$$

(2)

$$v(0, t) = v(L, t) = 0, \quad t \geq 0.$$

We assume the following:

(3) $$Q \in C^2((-\infty, \infty)) \text{ and } Q(\xi) > 0 \text{ for } \xi \in (-\infty, \infty),$$

(4) $$f, g \in C^2([0, L]), \ f(0) = f(L) = f''(0) = f''(L) = g'(0) = g'(L) = 0.$$

We extend f and g to be an odd periodic function and an even periodic function, respectively, with respect to $x = 0$, with periods $2L$. Under (4), $f, g \in C^2((-\infty, \infty))$.

Suppose $U = \begin{bmatrix} u \\ v \end{bmatrix}$ is a solution of (1), (2). By defining $u(x, t) = u(-x, t)$, $v(x, t) = -v(-x, t)$ for $-L \leq x \leq 0$, and $U(x + 2kL, t) = U(x, t)$ for $-L \leq x \leq L$, where k is any integer, the extended U is a solution of (1) with the initial conditions

$$u(x, 0) = g(x), \quad v(x, 0) = f(x) \quad (-\infty < x < \infty).$$

We define $r = v + M(u)$, $s = v - M(u)$, where $M(\xi) = \int_0^\xi Q(\eta)\, d\eta$. Then r and s are Riemann invariants of (1). Let $q(\eta) = Q(M^{-1}(\eta/2))$. The following three systems were derived in [5].

Under the transformation $R = \begin{bmatrix} r \\ s \end{bmatrix}$, (1) is transformed to

$$r_t(x, t) - q(r(x, t) - s(x, t))r_x(x, t) = 0,$$

(5)

$$s_t(x, t) + q(r(x, t) - s(x, t))s_x(x, t) = 0.$$

Under the hodograph transformation, (5) is transformed to

(6) $$\mathscr{L}X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{where } \mathscr{L} = \begin{bmatrix} \frac{\partial}{\partial r} & -q(r - s)\frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} & q(r - s)\frac{\partial}{\partial s} \end{bmatrix} \text{ and } X = \begin{bmatrix} x(r, s) \\ t(r, s) \end{bmatrix}.$$

Eliminating x in (6) gives

(7) $$t_{rs} = \rho(r - s)(t_r - t_s), \quad \text{where } \rho(\xi) = Q'(M^{-1}(\xi/2))/4q^2(\xi).$$

By (6), the Jacobian $\det \nabla_R X = 2qt_r t_s$.

LEMMA 1. *Suppose that* X(R) *is a solution of* (6) *in an open region* W *with the property that any two points in* W *can be connected by one of the following:*

(i) *a horizontal segment in* W,

(ii) *a vertical segment in* W,

(iii) *one or two line segments in* W *of positive slopes,*

(iv) *one or two line segments in* W *of negative slopes.*

If $t_r \cdot t_s \neq 0$ *in* W, *then* X(R) *is a homeomorphism on* W *and the image* D = X(W) *is an open region.*

The following lemmas are Lemmas 1 and 2 in [5].

LEMMA 2. *Suppose that* X(R) *is a solution of* (6). *If* W *is an open region contained in the domain of the function* X(R) *such that* X(R) *is one-to-one on* W *with nonvanishing Jacobian* $\det \nabla_R X = 2qt_r t_s$, *then*

$$U(X) = \begin{bmatrix} M^{-1}\left(\dfrac{r(x, t) - s(x, t)}{2}\right) \\ \\ \dfrac{r(x, t) + s(x, t)}{2} \end{bmatrix} \tag{8}$$

is a solution of (1) *on* D = X(W).

LEMMA 3. *If* X *satisfies the assumptions of Lemma* 2 *and is continuous on* $\overline{W}$, *and if* $t_r \to 0$ *or* $t_s \to 0$ *as* $R \to R_0 \in \partial W$, *then* $|u_x(X(R))| \to \infty$ *or* $|v_x(X(R))| \to \infty$ *as* $R \to R_0$.

Definition 1. If there is a point (x_0, t_0) such that a smooth solution U(x, t) of (1), (2) exists in $D = \{(x, t): 0 \leq x \leq L;\ 0 \leq t < t_0\}$, and if one of the derivatives u_x, v_x, u_t, v_t becomes unbounded as (x, t) in D tends to (x_0, t_0), then we say that the solution U(x, t) *breaks down* at (x_0, t_0).

2. OUTLINE OF RESULTS OF MacCAMY AND MIZEL

R. C. MacCamy and V. J. Mizel [5] considered (1), (2) under the further assumptions:

$$Q'(\xi) > 0 \text{ for } \xi < 0 \text{ and } Q'(\xi) < 0 \text{ for } \xi > 0, \tag{9}$$

$$g(x) \equiv 0,\ f(x) \text{ is concave over } [0, L]. \tag{10}$$

Let $a = \max_{0 \leq x \leq L} f(x) = f(b)$ $(0 < b < L)$. We denote the portion of f(x) over [-b, b] by $f_1(x)$ and that of f(x) over [b, 2L - b] by $f_2(x)$. We define $F_i(r) = f_i^{-1}(r)/2q(0)$ $(-a \leq r \leq a;\ i = 1, 2)$. Let $a_1 = M(\infty)$, $a_2 = -M(-\infty)$. We assume that $a < \min(a_1, a_2)$.

The following theorem is the combination of Lemmas 4 and 5 and Theorem 1 of [5]:

THEOREM 1. (i) *For* i = 1, 2, *the problems* (*see Figure* 1)

$$t^0_{irs} = \rho(r - s)(t^0_{ir} - t^0_{is}) \ \textit{in}\ \Omega_i = B'BB_i,$$

$$t^0_{ir}(r, r) = -t^0_{is}(r, r) = F'_i(r),\quad t^0_i(r, r) = 0, \quad -a \leq r \leq a, \tag{11}$$

together with (6) *and the initial conditions* $x^0_i(r, r) = 2q(0)\,F_i(r)$, $-a \leq r \leq a$, *provide a function* $X^0_i = \begin{bmatrix} x^0_i \\ t^0_i \end{bmatrix}$ *which is a homeomorphism on* Ω_i. *Moreover,* $t^0_{1r} > 0$

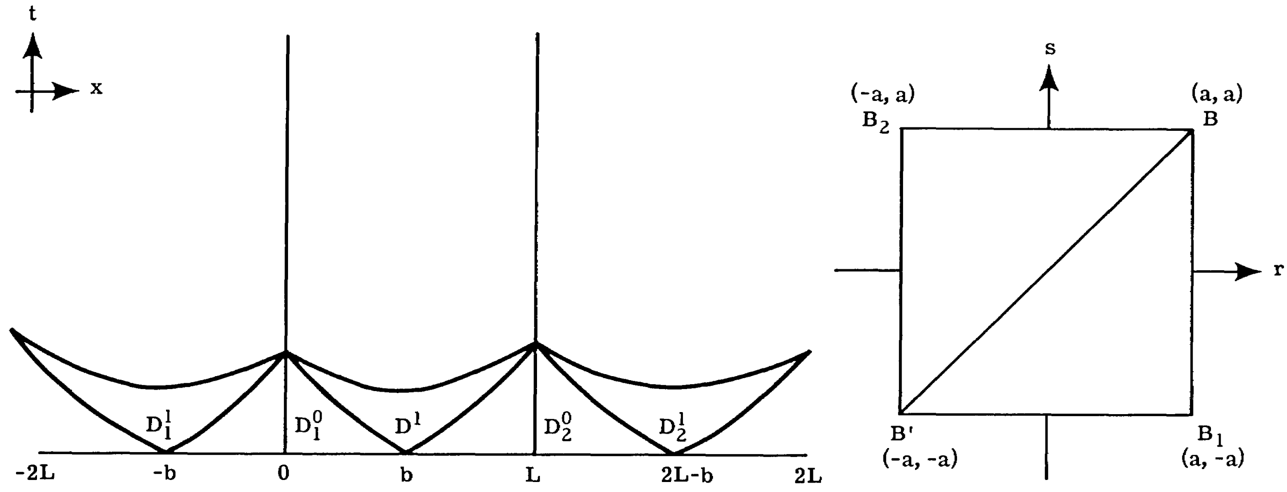

Figure 1.

and $t^0_{1s} < 0$ *in* Ω_1; $t^0_{2r} < 0$ *and* $t^0_{2s} > 0$ *in* Ω_2.

(ii) *The problem*

$$\mathscr{L} X^1 = 0 \ \textit{in} \ \Omega = \Omega_1 \cup \Omega_2,$$

$$(12) \qquad X^1(a, s) = X^0_1(a, s), \quad X^1(r, a) = X^0_2(r, a), \quad -a \leq r \leq a, \ -a \leq s \leq a,$$

provides a function $X^1 = \begin{bmatrix} x^1 \\ t^1 \end{bmatrix}$ *which is a homeomorphism on a neighborhood* Z *of* $r = a$ *and* $s = a$ *in* Ω. $t^1_r < 0$ *and* $t^1_s < 0$ *in* Z.

(iii) *The function* U *defined by*

$$U(X) = \begin{cases} U(X^0_i) & \textit{for} \ X \in D^0_i \quad (i = 1, 2), \\ U(X^1) & \textit{for} \ X \in D^1 \end{cases}$$

as in (8) *is a solution in* $D = D^0_1 \cup D^1 \cup D^0_2$ *of the system* (1) *with initial conditions* $u(x, 0) = 0$, $v(x, 0) = f(x)$ *for* $-b \leq x \leq 2L - b$, *where* $D^0_i = X^0_i(\Omega_i)$ *and* $D^1 = X^1(Z)$.

(iv) *The function* U(X) *satisfies* $u(-x, t) = u(x, t)$ *and* $v(-x, t) = -v(x, t)$ *for points* $(x, t) \in D$ *with the property that* $(-x, t) \in D$. U(X) *also satisfies*

$$u(2L - x, t) = u(x, t) \quad \textit{and} \quad v(2L - x, t) = -v(x, t)$$

for $(x, t) \in D$ *such that* $(2L - x, t) \in D$. U(X) *is a local solution of* (1), (2).

Constructing $X^1(r, s)$ as in Theorem 1(ii), we can solve a series of characteristic initial-value problems as described in section 3 of [5] and construct the functions X^{2n}_i, X^{2n+1}_i, and X^{2n+1} over Ω. The following recursion lemma is essentially Lemma 6 of [5].

LEMMA 4. *Let* X^i (i = 1, 2, 3, 4) *be the known solution of* (6) *in a square* W (*see Figure* 2) *with initial conditions along characteristic sides* [1, 2], [1, 4], [3, 2], *and* [3, 4] *of* W, *respectively, such that* $X^2 = X^1$ *along side* 4, $X^3 = X^1$ *along side* 3, $X^4 = X^2$ *along side* 3, *and* $X^4 = X^3$ *along side* 4. *Let the images* $D^i = X^i(W)$ *be as shown in Figure* 2. *Then*

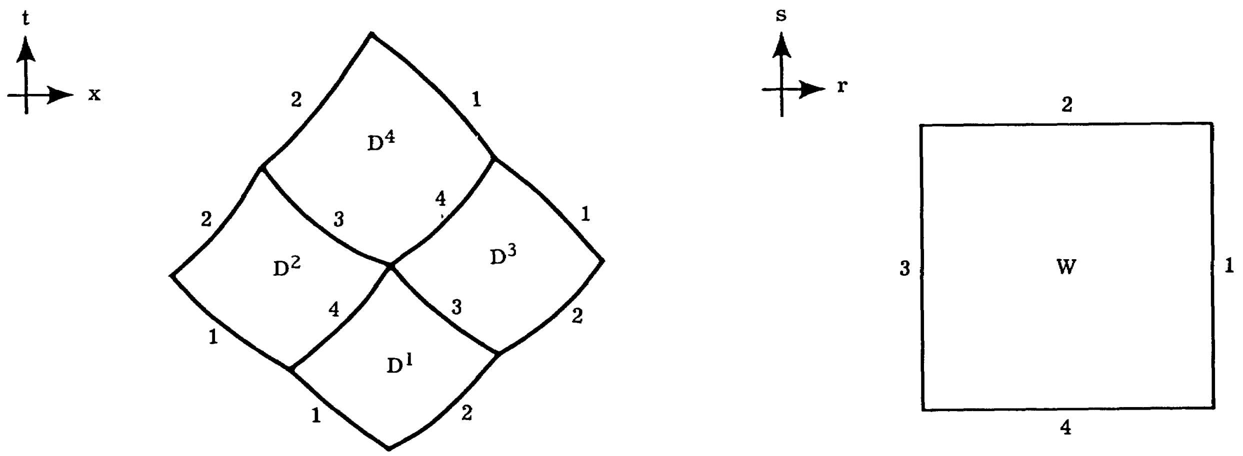

Figure 2.

$$x^4 = x^2 + x^3 - x^1 \quad \text{and} \quad t^4 = t^2 + t^3 - t^1 \quad \text{in } W.$$

Now f(x) is odd with respect to $x = 0$ and to $x = L$. We can show by the uniqueness theorems for (7) that for $n \geq 0$,

$$\text{(13)} \qquad \begin{aligned} t_i^{2n}(r, s) &= t_i^{2n}(-s, -r) \quad (i = 1, 2), \\ t_i^{2n+1}(r, s) &= t^{2n+1}(-s, -r) \quad (i = 1, 2). \end{aligned}$$

Let $\Psi(r, s) = t^1(r, s) + t^1(-s, -r) - t_1^0(r, s) - t_2^0(r, s)$. By (13), $\Psi(r, s) = \Psi(-s, -r)$. Applying Lemma 4 and (13), we obtain Lemma 8 of [5]:

LEMMA 5. *For* $k \geq 0$,

$$\begin{aligned} t^{4k+1}(r, s) &= 2k\Psi(r, s) + t^1(r, s), \\ t_1^{4k+2}(r, s) &= (2k+1)\Psi(r, s) + t_2^0(r, s), \\ t_2^{4k+2}(r, s) &= (2k+1)\Psi(r, s) + t_1^0(r, s), \\ t^{4k+3}(r, s) &= (2k+1)\Psi(r, s) + t^1(-s, -r), \\ t_1^{4k+4}(r, s) &= (2k+2)\Psi(r, s) + t_1^0(r, s), \\ t_2^{4k+4}(r, s) &= (2k+2)\Psi(r, s) + t_2^0(r, s). \end{aligned}$$

Remark 1. Lemma 4 was derived without assuming (9) and (10). Lemma 5 was derived without assuming (9).

Remark 2. Without assuming (9), we can prove a local existence theorem similar to Theorem 1. Indeed, we note that $t_{ir}^0(r, r) \cdot t_{is}^0(r, r) \neq 0$ and

$$t_r^1(a, a) \cdot t_s^1(a, a) \neq 0.$$

Then there exist neighborhoods N_i of $r = s$ in Ω_i and N of (a, a) in Ω which satisfy the assumptions on W of Lemma 1 for the functions t_i^0 and t^1, respectively. By Lemmas 1 and 2, and by the method used in showing Theorem 1(iv), we can show that U(X) constructed as in Theorem 1(iii) is a local solution of (1), (2).

Remark 3. By the methods introduced in [5] and in Remark 2, we can prove a more general local existence theorem under the assumptions (3) on Q and (4) and the following on f and g: $f'(x) + Q(g(x))\,g'(x)$ and $f'(x) - Q(g(x))\,g'(x)$ vanish on only two finite sets contained in $[0, L]$ respectively;

$$\max(M_1, M_2) < \min(a_1, a_2), \quad \text{where } M_1 = \max\{\,|f(x) + M(g(x))| : 0 \le x \le L\}$$
$$\text{and } M_2 = \max\{\,|f(x) - M(g(x))| : 0 \le x \le L\}\,.$$

The essential arguments are that the initial curve of (6) is the union of a finite number of strictly increasing or decreasing curves. Suppose $s = s(r)$ describes one of these curves. By setting $x(r, s(r)) = (f + M(g))^{-1}(r) = F(r)$ and using (6), we can set the initial conditions

$$t_r(r, s(r)) = -s'(r)\,t_s(r, s(r)) = F'(r)/2q(r - s(r)) \quad \text{and} \quad t(r, s(r)) = 0$$

for (7). Applying Lemmas 1 and 2, we can construct a local solution U of (1), (2) represented as in (8).

3. FINITENESS OF Ψ_r AND OF Ψ_s

We can write (7) as

$$(2\sqrt{q(r - s)}\,t_r(r, s))_s = q^*(r - s)\,t_s(r, s), \tag{14}$$

or

$$(2\sqrt{q(r - s)}\,t_s(r, s))_r = -q^*(r - s)\,t_r(r, s), \tag{15}$$

where $q^*(\xi) = -Q'(M^{-1}(\xi/2))/2q^{3/2}(\xi)$.

In sections 3 and 4, we assume (3), (4), and (10). As we observed in Remark 2, we can construct a local solution U of (1), (2) as described in Theorem 1. By (10) and (11),

$$t^0_{1r}(a, a) = -t^0_{1s}(a, a) = t^0_{1r}(-a, -a) = -t^0_{1s}(-a, -a) = \infty,$$
$$t^0_{2s}(a, a) = -t^0_{2r}(a, a) = t^0_{2s}(-a, -a) = -t^0_{2r}(-a, -a) = \infty.$$

Integrating (14) and (15) give

$$2\sqrt{q(r - s)}\,t^0_{ir}(r, s) = \int_r^s q^*(r - s')\,t^0_{is}(r, s')\,ds' + 2\sqrt{q(0)}\,t^0_{ir}(r, r), \tag{16}$$

$$2\sqrt{q(r - s)}\,t^0_{is}(r, s) = -\int_s^r q^*(r' - s)\,t^0_{ir}(r', s)\,dr' + 2\sqrt{q(0)}\,t^0_{is}(s, s). \tag{17}$$

Equations (16) and (17) form a system of Volterra integral equations. In $\mathring{\Omega}$, the interior of Ω, $t^0_{ir}(r, s)$ and $t^0_{is}(r, s)$ are finite. For $-a < r < a$, by (16) and by integration by parts,

$2\sqrt{q(r-a)}\, t^0_{ir}(r, a)$

$$= -\int_r^a (q^*(r-s'))_s\, t^0_i(r, s')\, ds' + q^*(r-a)\, t^0_i(r, a) + 2\sqrt{q(0)}\, t^0_{ir}(r, r)$$

is finite. That is, $t^0_{ir}(r, a)$ is finite for $-a < r < a$. Similarly, $t^0_{ir}(r, -a)$, $t^0_{is}(a, s)$, and $t^0_{is}(-a, s)$ are finite for $-a < r < a$ and $-a < s < a$.

We define

$$T^0_i(r, s) = 2\sqrt{q(r-s)}\, t^0_{ir}(r, s) - 2\sqrt{q(r-a)}\, t^0_{ir}(r, a)$$

for $(r, s) \in \mathring{\Omega}$, or for $-a < r < a$ and $-a \le s \le a$. We define

$$Y^0_i(r, s) = -2\sqrt{q(r-s)}\, t^0_{is}(r, s) + 2\sqrt{q(a-s)}\, t^0_{is}(a, s) \tag{18}$$

for $(r, s) \in \mathring{\Omega}$, or for $-a \le r \le a$ and $-a < s < a$.

By (16) and by integration by parts,

$$\lim_{r \to a} T^0_i(r, s) = \int_a^s q^*(a-s')\, t^0_{is}(a, s')\, ds'$$

is finite. We define $T^0_i(a, s)$ as this limit. We define $Y^0_i(r, a)$, $T^0_i(-a, s)$, and $Y^0_i(r, -a)$ similarly.

We have shown the following:

LEMMA 6. *In* $\mathring{\Omega}$, t^0_{ir} *and* t^0_{is} *are finite.* $t^0_{ir}(r, a)$, $t^0_{ir}(r, -a)$, $t^0_{is}(a, s)$, *and* $t^0_{is}(-a, s)$ *are finite for* $-a < r < a$ *and* $-a < s < a$. T^0_i *and* Y^0_i *are finite in* Ω *and*

$$\begin{aligned} t^0_{1r}(a, s) &= -t^0_{1s}(r, a) = t^0_{1r}(-a, s) = -t^0_{1s}(r, -a) = \infty, \\ t^0_{2s}(r, a) &= -t^0_{2r}(a, s) = t^0_{2s}(r, -a) = -t^0_{2r}(-a, s) = \infty. \end{aligned} \tag{19}$$

By (12), $t^1(a, s) = t^0_1(a, s)$ and $t^1(r, a) = t^0_2(r, a)$. We define

$$T^1(r, s) = 2\sqrt{q(r-s)}\, t^1_r(r, s) - 2\sqrt{q(r-a)}\, t^0_{2r}(r, a)$$

and

$$Y^1(r, s) = -2\sqrt{q(r-s)}\, t^1_s(r, s) + 2\sqrt{q(a-s)}\, t^0_{1s}(a, s). \tag{20}$$

By a method similar to that used in showing Lemma 6, we can show:

LEMMA 7. *In* $\mathring{\Omega}$, t^1_r *and* t^1_s *are finite.* $t^1_r(r, a)$, $t^1_r(r, -a)$, $t^1_s(a, s)$, *and* $t^1_s(-a, s)$ *are finite for* $-a < r < a$ *and* $-a < s < a$. T^1 *and* Y^1 *are finite in* Ω *and* $t^1_r(a, s) = t^1_s(r, a) = t^1_r(-a, s) = t^1_s(r, -a) = -\infty$.

Now

$$\lim_{r \to a} (t^1_r(r, s) - t^0_{2r}(r, s)) = \frac{1}{2\sqrt{q(a-s)}} (T^1(a, s) - T^0_2(a, s)).$$

By (13),

$$\lim_{r \to a} (-t_s^1(-s, -r) - t_{1r}^0(r, s)) = \frac{1}{2\sqrt{q(a-s)}} (Y^1(-s, -a) - Y_1^0(-s, -a)).$$

It follows from Lemmas 6 and 7 that

$$\Psi_r(a, s) = \frac{1}{2\sqrt{q(a-s)}} [T^1(a, s) - T_2^0(a, s) + Y^1(-s, -a) - Y_1^0(-s, -a)] \tag{21}$$

is finite. Similarly,

$$\Psi_s(r, a) = -\frac{1}{2\sqrt{q(r-a)}} [Y^1(r, a) - Y_1^0(r, a) + T^1(-a, -r) - T_2^0(-a, -r)]$$

is finite. By (13), $\Psi_r(r, s) = -\Psi_s(-s, -r)$. Thus $\Psi_s(r, -a)$ and $\Psi_r(-a, s)$ are finite.

We have shown the following:

LEMMA 8. *The function* Ψ, *which is the solution of*

$$\Psi_{rs} = \rho(r-s)(\Psi_r - \Psi_s) \text{ in } \Omega, \quad \Psi(a, s) = t^1(-s, -a) - t_2^0(a, s),$$

$$\Psi(r, a) = t^1(-a, -r) - t_1^0(r, a),$$

has finite first-order partial derivatives in Ω.

4. CONDITIONS FOR BREAKDOWN

THEOREM 2. *If* $\Psi \not\equiv$ *constant in* Ω, *then* U *breaks down.*

Proof. $\Psi \not\equiv$ constant implies that there is a point (r_0, s_0) in $\overset{o}{\Omega}$ such that either $\Psi_r(r_0, s_0) \neq 0$ or $\Psi_s(r_0, s_0) \neq 0$. Suppose $\Psi_r(r_0, s_0) > 0$. By Lemmas 5, 6, and 8,

$$0 < t_{2r}^{4k}(r_0, s_0) = 2k\Psi_r(r_0, s_0) + t_{2r}^0(r_0, s_0) < \infty$$

for sufficiently large k. By (19) and Lemma 8, $t_{2r}^{4k}(a, s) = 2k\Psi_r(a, s) + t_{2r}^0(a, s) = -\infty$. Thus t_{2r}^{4k} changes sign in Ω. By Lemma 3, U must break down.

Similarly, we can prove the breakdown of U for other cases.

THEOREM 3. *If* t_i^j $(i = 1, 2;\ j = 0, 2)$ *and* t^1 *satisfy one of the following three conditions:*

(i) $t_r^1 = 0$ *or* $t_s^1 = 0$ *somewhere in* Ω,

(ii) $t_{1r}^2 = 0$ *or* $t_{2r}^2 = 0$ *somewhere in* Ω,

(iii) $\Psi \not\equiv$ *constant in* Ω,

then U *breaks down. Conversely, if* U *breaks down, then one of the above three conditions holds.*

Proof. It follows from Lemma 3 and Theorem 2 that each of the conditions (i), (ii), (iii) is a sufficient condition for U to break down.

Suppose (i), (ii), and (iii) all fail to hold. Then $\Psi_r = \Psi_s \equiv 0$ in Ω, and by Lemmas 2 and 5 and by (13), we can extend U as a smooth function for all time over

$0 \leq x \leq L$. Thus if U does break down, then one of the conditions (i), (ii), (iii) must hold.

We define

$$A_1^- = \{\xi: 0 \leq \xi \leq M^{-1}(a) \text{ and } Q'(\xi) < 0\}, \quad A_1^+ = \{\xi: 0 \leq \xi \leq M^{-1}(a) \text{ and } Q'(\xi) > 0\},$$

$$A_2^- = \{\xi: M^{-1}(-a) \leq \xi \leq 0 \text{ and } Q'(\xi) < 0\}, \quad A_2^+ = \{\xi: M^{-1}(-a) \leq \xi \leq 0 \text{ and } Q'(\xi) > 0\}.$$

In the next theorem, the word "measure" means Lebesgue measure.

THEOREM 4. *If* Q' *satisfies one of the following four conditions:*

(i) $Q'(\xi) \leq 0$ *for* $0 \leq \xi \leq M^{-1}(a)$ *and* A_1^- *is of nonzero measure,*

(ii) $Q'(\xi) \geq 0$ *for* $0 \leq \xi \leq M^{-1}(a)$ *and* A_1^+ *is of nonzero measure,*

(iii) $Q'(\xi) \leq 0$ *for* $M^{-1}(-a) \leq \xi \leq 0$ *and* A_2^- *is of nonzero measure,*

(iv) $Q'(\xi) \geq 0$ *for* $M^{-1}(-a) \leq \xi \leq 0$ *and* A_2^+ *is of nonzero measure,*

then U *breaks down.*

Proof. Suppose that Q' satisfies (i). It suffices to assume that U can be extended to D. By Lemmas 2 and 3, t_{ir}^0, t_{is}^0, t_r^1, and t_s^1 maintain their initial signs. That is, $t_{1r}^0 > 0$ and $t_{1s}^0 < 0$ in Ω_1; $t_{2r}^0 < 0$ and $t_{2s}^0 > 0$ in Ω_2; $t_r^1 < 0$ and $t_s^1 < 0$ in Ω.

Now $T_2^0(a, a) = T^1(a, a) = 0$. By (15), (18), and (20),

$$-Y_1^0(-a, -a) = \int_{-a}^{a} q^*(r' + a)\, t_{1r}^0(r', -a)\, dr' > 0,$$

and

$$Y^1(-a, -a) = -\int_{-a}^{a} q^*(r' + a)\, t_r^1(r', -a)\, dr' > 0.$$

It follows from (21) that $\Psi_r(a, a) = \dfrac{1}{2\sqrt{q(0)}}\,(Y^1(-a, -a) - Y_1^0(-a, -a)) > 0$. Then $\Psi \not\equiv$ constant in Ω. By Theorem 2, U must break down.

Similarly, we can show that each of the conditions (ii), (iii), and (iv) is a sufficient condition for U to break down.

Remark 4. A statement equivalent to that of Theorem 2 is that a necessary condition for the existence of a smooth global solution U is that U be periodic in t in the sense that $\Psi \equiv$ constant. An example is the linear problem: $Q' \equiv 0$. For we know that in this case a smooth global solution U exists, which is periodic in t. By (12), (13), (14), and (15), $\Psi_r = \Psi_s \equiv 0$ in Ω. That is, $\Psi \equiv \omega$, a constant. By Lemma 5, 2ω is a period of U in t.

Remark 5. MacCamy and Mizel assumed essentially (i) and (iv) of Theorem 4 to derive the breakdown result (see Remark 10 of [5]). We showed that each one of these two conditions is a sufficient condition for U to break down.

5. SOME OTHER CASES

For $f(x) \equiv 0$, $g(x) = a\pi\cos \pi x$, and $Q^2(\xi) = (1 + \varepsilon\xi)^{\alpha}$, where α and ε are fixed positive numbers, N. J. Zabusky [6] has shown that the solution of (1), (2) breaks down. For general g, P. D. Lax [4] extended the breakdown result by assuming $Q'(\xi) \geq A > 0$.

We assume that $Q(\xi) > 0$ for $\xi \in (-d_2, d_1)$ and that $Q \in C^2((-d_2, d_1))$, where $d_i \in (0, \infty]$. We assume also that $f(x) \equiv 0$ and that $g \in C^2([0, L])$, $g'(0) = g'(L) = 0$, and the graph of g over $[0, L]$ consists of three parts: the first part is concave, the second part is convex, and the third part is concave. Let $g(0) = g(L) = \tilde{\tilde{a}}$. Define $\bar{b}$ $(0 < \bar{b} < L)$, by $g(\bar{b}) = \min_{0 \leq x \leq L} g(x) = \bar{\bar{a}}$. Define $\bar{a} = M(\bar{\bar{a}})$, $\tilde{a} = M(\tilde{\tilde{a}})$, $a_1 = M(d_1)$, $a_2 = -M(-d_2)$. We assume that $\max(|\bar{a}|, |\tilde{a}|) < \min(a_1, a_2)$.

By the methods introduced in Remark 3, we can construct a local solution U of (1), (2). By the method of extension in Theorem 1(ii), we can find subdomains of U as described in Figure 3. We define

$$\Phi(r, s) = t_1^1(r, s) + t_1^1(-s, -r) + t_0^1(r, s) + t_2^1(r, s) - t_1^0(r, s) - t_1^0(-s, -r)$$
$$- t_2^0(r, s) - t_2^0(-s, -r)$$

for $(r, s) \in \Omega$, where Ω is the square $\{(r, s): \bar{a} \leq r \leq \tilde{a};\ -\tilde{a} \leq s \leq -\bar{a}\}$.

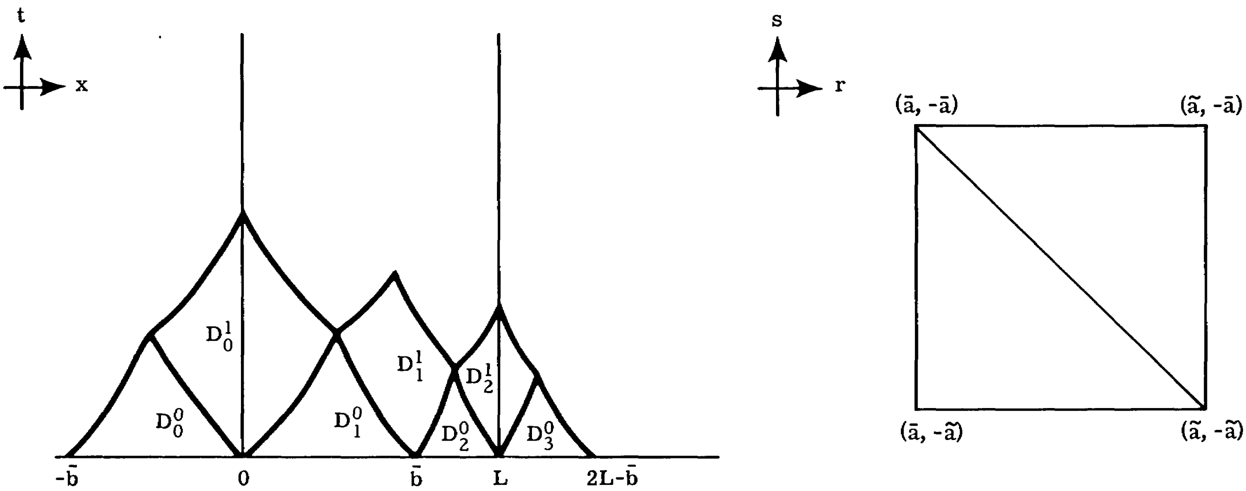

Figure 3.

By methods similar to those used in sections 3 and 4, we can show:

THEOREM 5. *If* $\Phi \not\equiv$ *constant in* Ω, *then* U *breaks down.*

THEOREM 6. *If* t_i^j *and* t_k^ℓ $(i = 1, 2;\ j = 0, 2;\ k = 0, 1, 2;\ \ell = 1, 3)$ *satisfy one of the three conditions:*

(i) $t^j_{ir} = 0$ *or* $t^j_{is} = 0$ *somewhere in* Ω *for some* i *and* j,

(ii) $t^\ell_{kr} = 0$ *or* $t^\ell_{ks} = 0$ *somewhere in* Ω *for some* k *and* ℓ,

(iii) $\Phi \not\equiv$ *constant in* Ω,

then U *breaks down. Conversely, if* U *breaks down, then one of the above three conditions holds.*

THEOREM 7. *Let*

$$A^+ = \{\xi: \bar{\bar{a}} \leq \xi \leq \tilde{\tilde{a}} \text{ and } Q'(\xi) > 0\} \quad \text{and} \quad A^- = \{\xi: \bar{\bar{a}} \leq \xi \leq \tilde{\tilde{a}} \text{ and } Q'(\xi) < 0\}.$$

If Q' *satisfies one of the conditions:*

(i) $Q'(\xi) \geq 0$ *for* $\bar{\bar{a}} \leq \xi \leq \tilde{\tilde{a}}$ *and* A^+ *is of nonzero measure,*

(ii) $Q'(\xi) \leq 0$ *for* $\bar{\bar{a}} \leq \xi \leq \tilde{\tilde{a}}$ *and* A^- *is of nonzero measure,*

then U *breaks down.*

We observe that (i) of Theorem 7 is weaker than the conditions assumed on Q' in [6] and [4].

For the problem considered in Remark 3, it is possible to derive sufficient conditions on Q' similar to those in Theorems 4 and 7 for the breakdown. An essential part of the derivations is to find recursion formulas for the functions t_i^j by a generalized version of Lemma 4. For brevity we do not pursue this problem here.

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