ON THE EULER CHARACTERISTIC OF REAL VARIETIES

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In this paper we give a bound for the Euler characteristic of algebraic varieties in real projective space. This provides a generalization of, and a conceptual framework for, the well-known work of Petrovsky and Oleĭnik [4] which estimates the Euler characteristic of a nonsingular hypersurface by means of a detailed analysis of the critical points of polynomials. Our result applies to any smooth variety. It is, in fact, much easier to state and prove the theorem in its general setting than to notice that it yields the more computationally formulated Petrovsky-Oleĭnik inequality when applied to hypersurfaces.

We obtain this topological information about the algebraic variety $V \subset \mathbb{R}P^N$ by comparing V with its *complexification* $V_C \subset \mathbb{C}P^N$. By definition V_C is the complex projective variety of all complex solutions to the polynomials which define V.

Note (*added in revision*). After this manuscript was prepared, the author learned that similar results have been obtained by Kharlamov in [2]. In that paper he proves Theorem 1 and states Proposition 1 as a conjecture.

THEOREM 1. *Let* $V^{2n} \subset \mathbb{R}P^N$ *be a nonsingular projective real algebraic variety and suppose that* V_C, *the complexification of* V, *is also smooth. Then the Euler characteristic of* V *is bounded by* $|\chi(V)| \leq \dim H^{n,n}(V_C)$.

Remark. We may always assume that V_C is smooth, since a small variation of the defining polynomials eliminates any singularities of V_C while altering V by a diffeomorphism. A smooth V_C admits a Hodge decomposition of its complex cohomology [see 6]

$$H^k(V_C; \mathbb{C}) = \sum_{p+q=k} H^{p,q}(V_C),$$

and the right hand side of the above inequality refers to this decomposition.

Proof of Theorem 1. Complex conjugation $T: \mathbb{C}P^N \to \mathbb{C}P^N$,

$$T(z_0, z_1, \cdots, z_N) = (\bar{z}_0, \bar{z}_1, \cdots, \bar{z}_N),$$

restricts to an involution of V_C with V as fixed-point set. Moreover, T is an isometry, and therefore the Euler characteristic of V is equal to the Lefschetz number $L(T, V_C)$. (See [3], p. 76.)

Let J be the almost complex structure on the tangent bundle of V_C. T carries J to the "conjugate" structure $-J$. Acting on forms, then, T^* sends forms of type (p, q) to forms of type (q, p). Therefore, for every p and q, T^* preserves the direct sum $H^{p,q}(V_C) \oplus H^{q,p}(V_C)$. For $p \neq q$, the trace of T^* restricted to this sum is zero. Therefore

$$L(T, V_C) = \sum_{j=0}^{4n} (-1)^j \operatorname{tr}(T^* \mid H^j(V_C)) = \operatorname{tr}(T^* \mid W),$$

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where $W = \sum_{k=0}^{2n} H^{k,k}(V_C)$.

Let $\omega \in H^{1,1}(V_C)$ be the Kähler class. Splitting each $H^{k,k}(V_C)$ according to the Lefschetz decomposition [6], we can rewrite W as

$$W = \sum_{k=0}^{n} (P^{k,k} \oplus \omega P^{k,k} \oplus \cdots \oplus \omega^{2(n-k)} P^{k,k}),$$

where $P^{k,k}$ is the primitive part of $H^{k,k}(V_C)$. Since T sends J to $-J$, one sees that $T^*(\omega) = -\omega$. Hence T^* preserves the Lefschetz decomposition and $\operatorname{tr}(T^* \mid \omega^i P^{k,k}) = (-1)^i \operatorname{tr}(T^* \mid P^{k,k})$. This implies that

$$\operatorname{tr}(T^* \mid W) = \operatorname{tr}(T^* \mid \sum_{k=0}^{n} P^{k,k}).$$

The theorem now follows, since $\dim \sum_{k=0}^{n} P^{k,k} = \dim H^{n,n}$.

Remark. This recovery of topological restrictions on V coming from V_C is analogous to the observation of Thom [5] that the sum of the mod 2 Betti numbers of V is bounded by the corresponding sum for V_C.

Suppose now that $V^{2n} \subset \mathbb{R}\mathbb{P}^{2n+r}$ is a smooth complete intersection of hypersurfaces of degrees $a_1, \cdots, a_r$. We can assume that V_C is likewise a complete intersection. By the Hirzebruch-Riemann-Roch theorem [1], $\dim H^{n,n}(V_C)$ is equal to $(-1)^n$ times the coefficient of $y^n z^{2n+r}$ in the power series expansion of

$$\frac{1}{(1-z)(1+zy)} \prod_{i=1}^{r} \frac{(1+zy)^{a_i} - (1-z)^{a_i}}{(1+zy)^{a_i} - y(1-z)^{a_i}}. \tag{1}$$

Our theorem thus gives an explicit bound for $\chi(V)$ in terms of the a_i.

In the case where $V^{2n} \subset \mathbb{R}\mathbb{P}^{2n+1}$ is a nonsingular hypersurface defined by a polynomial f of degree a, we can compare our result with that given by Petrovsky and Oleĭnik [4].

PETROVSKY-OLEĬNIK INEQUALITY. *If* $V \subset \mathbb{R}\mathbb{P}^{2n+1}$ *is a smooth hypersurface of degree* a, *then*

$$|\chi(V)| \leq (a-1)^{2n+1} - 2S(2n+1, a) + 1, \tag{2}$$

where $S(2n+1, a)$ *is equal to the number of terms of degree less than or equal to* $an - (2n+1)$ *in the expansion of*

$$\prod_{i=1}^{2n+1} \frac{x_i^{a-1} - 1}{x_i - 1}. \tag{3}$$

The following proposition shows that this formidable expression is nothing more than $\dim H^{n,n}(V_C)$. Hence our theorem can be regarded as a generalization of the Petrovsky-Oleĭnik inequality, valid for any smooth real algebraic variety.

PROPOSITION 1. *The expression* (2) *is equal to* $\dim H^{n,n}(V_C)$, *where* $V_C \subset \mathbb{C}\mathbb{P}^{2n+1}$ *is a smooth hypersurface of degree* a.

Proof. By the Riemann-Roch formula (1), dim $H^{n,n}(V_C)$ is given by the coefficient of $y^n z^{2n+1}$ in the expansion of

$$F(y, z) = \frac{1}{(1-z)(1-zy)} \frac{(1-zy)^a - (1-z)^a}{(1-zy)^a - y(1-z)^a} .$$

To compare this with (2) we construct a generating function for the combinatorial expression $S(2n+1, a)$.

LEMMA 1. $S(2n+1, a)$ *is equal to minus the coefficient of* $y^n z^{2n+1}$ *in the power series expansion of*

$$G(y, z) = \frac{(1-zy)^{a-1}}{(1-zy)^a - y(1-z)^a} .$$

The proposition now reduces to showing that the coefficient of $y^n z^{2n+1}$ in $F(y, z) - 2G(y, z)$ is $(a-1)^{2n+1} + 1$. Since $F - 2G$ can be written as

$$\frac{z}{(1-z)(1-zy)} + \left(\frac{-1}{1-zy}\right)\left(\frac{1 + \left(\frac{1-z}{1-zy}\right)^{a-1}}{1 - y\left(\frac{1-z}{1-zy}\right)^{a}}\right) \tag{4}$$

and the coefficient of $y^n z^{2n+1}$ in the first term of (4) is 1, we need only show that the coefficient in the second term is $(a-1)^{2n+1}$. It is not hard to see that this reduces to the following identity in binomial coefficients.

LEMMA 2.

$$\sum_{k=0}^{n} (-1)^{n+k} \left[\binom{ak+n-k}{ak}\binom{ak}{n+k+1} + \binom{ak+a+n-k-1}{ak+a-1}\binom{ak+a-1}{n+k+1} \right] = (a-1)^{2n+1} . \tag{5}$$

Remark. The above proposition was initially conjectured after a number of low-dimensional cases had been conveniently verified with the aid of the Macsyma system for symbolic computation. I am grateful to Joel Moses of M.I.T.'s Project Mac for making the facilities available.

Proof of Lemma 1. Let $R(k, j)$ be the number of terms of degree less than or equal to j in

$$\prod_{i=1}^{k} \frac{x_i^{a-1} - 1}{x_i - 1} .$$

Let $T(k, j)$ be the number of terms of degree exactly j in the above expression. Then $T(k, j)$ is the coefficient of x^j in the power series expansion of Q^k, where $Q(x) = (x^{a-1} - 1)/(x - 1)$, and hence

$$\frac{Q^k}{1-x} = \sum_{j=0}^{\infty} x^j R(k, j) .$$

This shows that $S(2n+1, a)$, which is $R(2n+1, an-(2n+1))$, equals the coefficient of $x^{an}z^{2n+1}$ in

$$\frac{1}{(1+zx^a)-x(1+z)}, \tag{6}$$

because (6) is equal to

$$\frac{1}{(1-x)(1-zxQ)} = \sum_{k=0}^{\infty} z^k x^k \frac{Q^k}{1-x} = \sum_{k=0}^{\infty}\sum_{p=k}^{\infty} z^k x^p R(k, p-k) .$$

Changing z to $-z$ in (6) gives $-S(2n+1, a)$ as the coefficient of $x^{an}z^{2n+1}$ in

$$H(x, z) = \frac{1}{(1-zx^a)-x(1-z)},$$

which is the same as the coefficient of $x^{an}z^{2n+1}$ in

$$\frac{1}{a}\sum_{j=0}^{a-1} H(\eta^j x, z) = \frac{(1-zx^a)^{a-1}}{(1-zx^a)^a - x^a(1-z)^a}. \tag{7}$$

(Here η is a primitive ath root of unity.) Finally, we set $y = x^a$ in (7) and Lemma 1 is proved.

Proof of Lemma 2. Rewrite the summation in (5) as

$$\sum_{k=1}^{n+1} (-1)^{n+k}\left[\binom{ak+n-k}{ak}\binom{ak}{n+k+1} - \binom{ak+n-k}{ak-1}\binom{ak-1}{n+k}\right].$$

This simplifies to

$$\begin{aligned} &\frac{2}{2n+2}\sum_{k=0}^{n+1} (-1)^k (n+1-k)\binom{2n+2}{k}\binom{(n+1-k)(a-1)+n}{2n+1} \\ &= \frac{1}{2n+2}\sum_{k=0}^{2n+2} (-1)^k (n+1-k)\binom{2n+2}{k}\binom{(n+1-k)(a-1)+n}{2n+1}. \end{aligned} \tag{8}$$

Now we use the following fact: If $P(k)$ is any polynomial of degree $2n+2$, then the sum

$$\sum_{k=0}^{2n+2} (-1)^k \binom{2n+2}{k} P(k)$$

is equal to $(2n+2)!$ times the leading coefficient of P. Applying this to (8) with

$$P(k) = (n+1-k)\binom{(n+1-k)(a-1)+n}{2n+1}$$

completes the proof.

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