FRAMED MANIFOLDS WITH A FIXED POINT FREE INVOLUTION

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The aim of this note is to prove that every framed cobordism class of positive dimension can be represented by a framed manifold with a fixed point free involution which preserves the framing. In the following, all manifolds and maps are smooth wherever this makes sense.

Suppose M is a closed, compact m-manifold with a fixed point free involution t, ν_M is the normal bundle of $M \subset R^{m+k}$ (k large), and f: $\nu_M \to R^k$ is a framing. We say that t *preserves* f if the following condition is satisfied. Let N = M/t and let p: M → N be the projection. Then $M \to N \subset R^{m+k}$ is an immersion and hence p is covered by a canonical map s: $\nu_M \to \nu_N$ which is unique up to homotopy. We say that t preserves f if f = gs, where g: $\nu_N \to R^k$ is a framing. Let (M, f)/t = (N, g). Let Ω_*^{fr} denote the framed cobordism group and ${}_2\Omega_*^{fr}$ its two-primary part.

We prove the following theorem.

THEOREM 1. *If* $\alpha \in \Omega_m^{fr}$ (m > 0), *then* α *can be represented by* (M, f), *where* M *admits a fixed point free involution* t *which preserves* f. *If* $\alpha \neq 0$ *and* $\alpha \in {}_2\Omega_m^{fr}$, *then* (M, f) *and* t *can be chosen so that* M *is* [m/2]-*connected and* (M, f)/t *is framed cobordant to zero.*

We begin the proof of Theorem 1 by stating and proving a result of N. Ray [2]. Let P^{k-1} be real projective (k - 1)-space, let A: $R^k \to R^k$ be given by

$$A(x_1, x_2, \cdots, x_k) = (-x_1, x_2, \cdots, x_k),$$

and let λ: $P^{k-1} \to SO_k$ be the composition of A and the map which assigns to each line ℓ, the reflection through the orthogonal complement of ℓ. If g: $\nu_N \to R^k$ is a framing and u: $N \to P^{k-1}$, let ug: $\nu_N \to R^k$ be the framing given by

$$ug(v) = (\lambda up(v))(g(v)),$$

where p: $\nu_N \to N$ is the projection.

THEOREM 2 (N. Ray). *If* $\alpha \in {}_2\Omega_m^{fr}$ ($m > 0$, $\alpha \neq 0$), *then* α *can be represented by* (N, ug), *where* (N, g) *is framed cobordant to zero and* u_*: $\pi_i(N) \to \pi_i(P^{k-1})$ *is an isomorphism for* 2i < m.

Proof. Let $T(\nu_N)$ be the Thom space of ν_N, that is, the disc bundle modulo the sphere bundle, and let t: $S^{m+k} \to T(\nu_N)$ be the Thom-Pontrjagin construction. We identify Ω_m^{fr} with $\pi_{m+k}(S^k)$ under the map $\{N, g\} \to [T(g)t]$.

Let D^k be the unit k-disc and $S^{k-1} \circ P^{k-1}$ be $D^k \times P^{k-1}$ modulo the relation $(x, y) \approx (x, y')$ for $x \in S^{k-1}$. Let J: $S^{k-1} \circ P^{k-1} \to S^k = D^k/S^{k-1}$ be given by $J(x, y) = \lambda(y)(x)$. D. S. Kahn and S. B. Priddy [1] have shown that

Received July 14, 1975.
The author is supported by NSF Grant GP-38920X1.

Michigan Math. J. 23 (1976).

$$J_*: \pi_m(S^{k-1} \circ P^{k-1}) \to {}_2\pi_m(S^k)$$

is an epimorphism for k large with respect to m. Suppose $\alpha \in {}_2\pi_m(S^k)$ and $J_*(\beta) = \alpha$. By transverse regularity, we may represent β by

$$S^{m+k} \xrightarrow{t} T(\nu_N) \xrightarrow{h} S^{k-1} \circ P^{k-1},$$

where $h(v) = \{g(v), up(v)\}$, v is in the disc bundle of ν_N, $g: \nu_N \to R^k$, and $u: N \to P^{k-1}$. Note that $Jh = T(ug)$. Hence $T(ug)t = Jht \in J_*\beta = \alpha$. Thus (N, ug) represents α. But $T(g)t$ factors:

$$S^{m+k} \to T(\nu_N) \to S^{k-1} \circ P^{k-1} \to D^k \to S^k.$$

Hence, $T(g)t$ is homotopic to zero, and therefore (N, g) is cobordant to zero. Applying surgery to the commutative diagram

$$\begin{array}{ccc} \nu_N & \longrightarrow & R^k \times P^{k-1} \\ \downarrow & & \downarrow \\ N & \longrightarrow & P^{k-1} \end{array}$$

we may make $u_*: \pi_i(N) \to \pi_i(P^{k-1})$ an injection for $2i < m$. If $\alpha \neq 0$, then u must be nontrivial and u_* an isomorphism on π_1. This completes the proof of Theorem 2.

THEOREM 3. *Suppose* N *is a compact manifold and* $u: N \to P^{k-1}$. *There is a function* $F: N \times R^k \times [0, 1] \to S^k = R^k \cup \{\infty\}$ *satisfying:*

(i) F *is transverse regular to* 0. $F^{-1}(0) \cap N \times R^k \times \{1\} = N \times \{\pm e_1\} \times \{1\}$, *where* $e_1 = (1, 0, \cdots, 0)$. $F^{-1}(0) \cap N \times R^k \times \{0\} = M \times \{0\}$, *where* M *is the two-sheeted cover of* N, $\{(x, y) \in N \times S^{k-1}: y \in u(x)\}$.

(ii) *Identify* R^k *with the tangent vectors of* $N \times R^k \times [0, 1]$ *at* (n, x, t) *which are tangent to* $\{n\} \times R^k \times \{t\}$. *For* $x \in R^k$,

$$dF_z(x) = \begin{cases} x & \textit{if } z \in M \times \{0\} \cup N \times \{-e_1\} \times \{1\} \\ \lambda u(n)x & \textit{if } z = (n, e_1, 1) \textit{ and } n \in N. \end{cases}$$

Before proving Theorem 3, we show that it implies Theorem 1. Suppose $\alpha \in \Omega_m^{fr}$ $(m > 0)$. If α has odd order and $(N, g) \in \alpha$, then $2(N, g) \in \alpha$ and $2(N, g)$ has an obvious fixed point free involution. Suppose $\alpha \in {}_2\Omega_m^{fr}$. By Theorem 2, $\alpha = \{N, ug\}$, where $(N, g) \sim 0$. Let G be the composition

$$\nu_N \times [0, 1] \xrightarrow{p \times g \times id} N \times R^k \times [0, 1] \xrightarrow{F} S^k.$$

By Theorem 3, $G^{-1}(0) \subset R^{m+k} \times [0, 1]$ is a framed cobordism between (M, f) and (N, g) + (N, ug), where $f = sg$ and $s: \nu_M \to \nu_N$ is as above; this result was suggested to me by Jerome Levine. The covering translation t of M gives a fixed point free involution preserving f. Hence $(M, f)/t = (N, g) \sim 0$. Choosing N and u so that $u_*: \pi_i(N) \approx \pi_i(P^{k-1})$ $(2i < m)$, we see that M is [m/2]-connected.

Proof of Theorem 3. Note that it is sufficient to construct an F transverse regular to 0 on $N \times R^k \times \{0, 1\}$ and satisfying (i) and (ii). We construct F in two steps. First, for $N = P^1$, u = identity (F is the map G below), and then the general case.

Let C be the field of complex numbers and $S^1 = \{z \in C: |z| = 1\}$. Let

$$G: S^1 \times C \times [0, 1] \to S^2 = C \cup \{\infty\}$$

be defined by

$$G(z, w, t) = \frac{w^2 - z + t(z - 1)}{(2 + t(z - 1))\, w + t(z - 1)} .$$

Let N and D denote the numerator and denominator of the above fraction. To show that G is well defined, we must show that N and D do not vanish simultaneously. Suppose $\Im z \geq 0$. One may easily check that if $N = 0$, then $(\Im w)(\Re w) \geq 0$; and if $D = 0$, then $\Im w \leq 0$ and $\Re w \geq 0$. Hence, w is real if $N = D = 0$. The same argument, for $\Im z \leq 0$, also yields the conclusion that w is real. Obvious considerations show that D and N cannot both be zero when w is real. Observe that

$$\text{(1)} \qquad \begin{aligned} G^{-1}(0) \cap S^1 \times C \times \{0\} &= \{(z, w): w^2 = z\} \times \{0\}, \\ G^{-1}(0) \cap S^1 \times C \times \{1\} &= \{(z, \pm 1): z \in S^1\} \times \{1\}; \end{aligned}$$

$$\text{(2)} \qquad \frac{\partial G}{\partial w}(z, w, t) = \begin{cases} 1 & \text{if } w^2 = z \text{ and } t = 0 \\ & \quad \text{or} \\ & w = -1 \text{ and } t = 1 \\ \bar{z} & \text{if } w = 1 \text{ and } t = 1 . \end{cases}$$

Suppose $u: N \to P^{k-1}$. By increasing k if necessary, we may assume $e_1 \notin u(N)$. Suppose $n \in N$, $x \in S^{k-1}$ and $x \in u(n)$, $y \in R^k$, and $t \in [0, 1]$. Choose $r, s \in R$ and $x_1 \in S^{k-1}$ such that $(x_1 \cdot e_1) = 0$ and $x = re_1 + sx_1$. Since $x \neq e_1$, it follows that x_1 is unique up to sign. Let $g: C \to R^k$ be defined by $g(a + bi) = ae_1 + bx_1$, and let $y_1 = y - (y \cdot e_1)e_1 - (y \cdot x_1)x_1$. Define $F: N \times R^k \times [0, 1] \to S^k = R^k \cup \{\infty\}$ by

$$F(n, y, t) = gG((r + si)^2, (y \cdot e_1) + (y \cdot x_1)i, t) + y_1 .$$

One easily checks that the above is independent of the choice of r, s, and x_1, using the fact that $G(\overline{z}, \overline{w}, t) = \overline{G(z, w, t)}$.

By (1), $F^{-1}(0) \cap N \times R^k \times \{0, 1\} = M \times \{0\} \cup N \times \{\pm e_1\} \times \{1\}$. If n, x, and y are as above and $v = (n, e_1, 1)$, then by (2)

$$dF_v(y) = g(\overline{(r + si)}^2((y \cdot e_1) + (y \cdot x_1)i)) .$$

If $z \in S^1$, the mapping given by $w \to \bar{z}^2 w$ is the reflection through the line perpendicular to z composed with reflection through the imaginary axis. Thus dF_v is reflection through the orthogonal complement of the e_1 axis; that is, $dF_v = \lambda u(n)$. Similarly by (2), $dF_v(y) = y$, for $v \in M \times \{0\} \cup N \times \{-e_1\} \times \{1\}$. Thus F satisfies condition (ii), and F is transverse regular to 0 on $F^{-1}(0) \cap N \times R^k \times \{0, 1\}$. This completes the proof of Theorem 3.

REFERENCES

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