COMPACTIFICATION OF COVERING SPACES OF COMPACT 3-MANIFOLDS

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The problem of determining conditions under which a noncompact 3-manifold $\tilde{M}$ permits a *manifold compactification*, that is, an embedding $h: \tilde{M} \to Q$, where Q is compact and $h(\text{int } \tilde{M}) = \text{int } Q$, has recently received considerable attention (see [6], [2], [15], and references therein). Since h is a homotopy equivalence, $\tilde{M}$ must have a legal homotopy type: if $\tilde{M}$ contains no 2-sided projective plane, then, by Theorem 2.1 of [14] and Theorem 3.2 of [4], the assumption that

$$\pi_1(\tilde{M}) \quad \text{and} \quad \text{image}(\pi_2(\tilde{M}) \to H_2(\tilde{M}; \mathbb{Z}))$$

are finitely generated is sufficient to guarantee that $\tilde{M}$ has the homotopy type of a compact 3-manifold. Also, Kneser's theorem [11] and the unresolved Poincaré conjecture force us to require that $\tilde{M}$ contain at most finitely many fake 3-cells. But these conditions are far from sufficient: there exist contractible manifolds $\tilde{M} \subseteq \mathbb{R}^3$ that are open [13] or have interior $\mathbb{R}^3$ and boundary $\mathbb{R}^2$ [6, p. 230] but have no manifold compactifications. The obstruction, a sort of homotopic wildness at infinity, is studied in [6], [2], and [15].

This paper considers the question of the existence of a manifold compactification primarily in the case where $\tilde{M}$ is a covering space of a compact 3-manifold. Postulating P^2-irreducibility to eliminate problems with π_2, projective planes, and fake 3-cells, and hoping that the regularities inherent even in irregular covering spaces prevent wildness of π_1 at infinity, we are led to the following conjecture.

If M *is a* P^2*-irreducible, compact, connected* 3*-manifold and* H *is a finitely generated subgroup of* $\pi_1(M)$, *then the covering space* $\tilde{M}(H)$ *of* M *corresponding to* H *has a manifold compactification.*

In Section 2, we consider the technical problems involved in pasting together compactifications of manifolds to obtain a compactification of their union. The compactification theorem developed there is the basis for Section 3, in which we consider the problem of compactifying covering spaces.

In Corollary 3.3, we establish the conjecture on compactifying covering spaces for the cases where M is a line bundle over a 2-manifold or the product of a 2-manifold with S^1. In addition, we show in Corollary 3.2 that the class of manifolds for which the conjecture holds is closed under the operations of pasting along disks or along incompressible annuli, Möbius bands, and tori. In Theorem 3.7, we expand the class of well-behaved manifolds by restricting H: if M is a compact, connected, P^2-irreducible, *sufficiently large* manifold and the intersection of H with each finitely generated subgroup of $\pi_1(M)$ is finitely generated, then $\tilde{M}(H)$ has a manifold compactification. In particular, we indicate in Corollary 3.8 that Theorem 3.7 can be applied whenever H is abelian or is a subgroup of $\text{image}(\pi_1(\partial M) \to \pi_1(M))$.

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From this, it follows (Corollary 3.9) that if M is compact and P^2-irreducible and B is an incompressible boundary component of M, then $B \times [0, 1]$ is a manifold compactification of $\tilde{M}(\pi_1(B))$; similarly, $\tilde{M}(\mathbb{Z})$ embeds in a solid torus or solid Klein bottle, and if M is orientable, then $\tilde{M}(\mathbb{Z} \oplus \mathbb{Z})$ embeds in $S^1 \times S^1 \times [0, 1]$.

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1. PRELIMINARIES

We use the terms *manifold* and *surface* without tacit assumptions as to compactness, connectedness, orientability, or possession of boundary. The terms *incompressible* 2-manifold, *sufficiently large* 3-manifold, and P^2-*irreducible* 3-manifold have their usual meanings (see [4], or [18] and [5]). In particular, if M contains a compact, properly embedded, 2-sided, incompressible surface and M is not a 3-cell, M is called *sufficiently large.* If N is connected and $N \subseteq M$, we denote the inclusion-induced homomorphism of $\pi_1(N, x)$ into $\pi_1(M, x)$ by $(\pi_1(N) \to \pi_1(M))$. Lemmas 1.1 to 1.3 are well known facts which we collect here for future reference.

LEMMA 1.1. *If* M *is a connected,* P^2-*irreducible* 3-*manifold then* $\pi_2(M) = 0$.

Proof. [3, Section 1].

LEMMA 1.2. *Let* M *be a compact,* P^2-*irreducible* 3-*manifold satisfying one or more of the following conditions:*

(i) $\partial M \neq \emptyset$,

(ii) M *is nonorientable,*

(iii) $H_1(M; \mathbb{Z})$ *is infinite,*

(iv) $\pi_1(M)$ *is a nontrivial amalgamated free product.*

Then M *is sufficiently large.*

Proof. [5, Theorem 3] and [17, Satz 1.2].

LEMMA 1.3. *If* M *is a compact,* P^2-*irreducible, sufficiently large* 3-*manifold, then each covering space of* M *is* P^2-*irreducible.*

Proof. [5, Theorem 3], [18, Theorem 8.1], and [3, Sections 1 and 6].

2. A GENERAL COMPACTIFICATION THEOREM

This section is, in part, a generalization of the proof of Theorem 8.1 of [18].

Let M be a connected 3-manifold such that $\pi_1(M)$ is finitely generated. If M can be expressed as the union of submanifolds M_i such that each M_i has a manifold compactification $Q(M_i)$ and the M_i intersect nicely, we can try to construct a manifold compactification of M using the $Q(M_i)$ as building blocks. We accomplish this in Theorem 2.5.

The main difficulty we encounter is illustrated by the following example: let $\hat{M}_1$ and $\hat{M}_2$ be 3-cells; let A_1 be a point in $\partial\hat{M}_1$ and let A_2 be a closed disk in $\partial\hat{M}_2$; let $M_i = \hat{M}_i - A_i$ (i = 1, 2) and let M be the manifold obtained by pasting M_1 and M_2 together along their homeomorphic boundaries. Although M is homeomorphic to an open 3-cell, we cannot naively paste together the compactifications $\hat{M}_i$ of M_i (i = 1, 2) to form a compactification of M, since the closures of ∂M_1 in $\hat{M}_1$ and ∂M_2 in $\hat{M}_2$ are not homeomorphic. In Lemma 2.4, we show how to change the embedding of M_1 in $\hat{M}_1$ so that the closure of ∂M_1 in $\hat{M}_1$ is a closed disk. We can then paste together the manifolds $\hat{M}_i$ along the closures of ∂M_i to obtain a compact manifold Q. In this example, Q is a compactification of M regardless of our choice of attaching homeomorphism. In general, however, we must be able to find an attaching map for the compact manifolds that is compatible with the given attaching homeomorphism of the noncompact manifolds. Lemmas 2.2, 2.3, and condition (iii) of 2.4 guarantee this ability.

Definition. Let F be a 2-manifold and let G be a compact 2-manifold in the interior of F. If the pair (int F - int G, ∂G) is homeomorphic to

$$(\partial G \times [0, 1), \partial G \times \{0\}),$$

then we call G a *compact core* of F.

PROPOSITION 2.1. *A 2-manifold* F *has a compact core if and only if* F *has finitely many components and the fundamental group of each component of* F *is finitely generated. Also, if* G_1 *and* G_2 *are compact cores of* F, *then there exists a self-homeomorphism* g *of* F *such that* $g(G_1) = G_2$ *and* g *is the identity outside of some compact set.*

Definition. Let H be a compact 2-manifold. Let F be a 2-manifold contained in H satisfying the following conditions:

(i) int F = int H,

(ii) for each component J of ∂H, either $J \cap \partial F = \emptyset$ or $J \cap \partial F$ is dense in J.

Then we say F is *standardly embedded* in H.

The following lemma, which indicates the importance of the above property, will be used in the proof of Theorem 2.5.

LEMMA 2.2. *Let* F *be a standardly embedded submanifold of a compact* 2-*manifold* H. *If* $f: F \to F$ *is a homeomorphism, then there exists a homeomorphism* $g: F \to F$ *such that* g *is homotopic to* f *and* g *extends to a homeomorphism of* H *onto* H.

Proof. Let G be a compact core of F and let $C_1, \cdots, C_n$ be the components of H - int G. The manifold f(G) is a compact core of F and of H. Let $D_1, \cdots, D_n$ be the components of H - int f(G), ordered so that $f(F \cap C_i) = F \cap D_i$ (i = 1, ⋯, n). We define a homeomorphism $h: H \to H$ as follows. On the set G, let h = f. On each component C_i that contains points of ∂F, let h be the unique extension of the restriction $f|F \cap C_i$ to a homeomorphism of C_i onto D_i. This extension exists because F is standardly embedded in H. Finally, we define h on each component C_i that is disjoint from ∂F by letting h be any extension of the restriction $f|\partial G \cap C_i$ to a homeomorphism of C_i onto D_i. Since int F = int H, each such extension maps $F \cap C_i$ onto $F \cap D_i$. Thus the restriction $h|F$, which we denote by g, is a homeomorphism of F onto F. Furthermore, since $g|G = f|G$, g is homotopic to f.

In order to take advantage of the above lemma, we need to be able to produce standard embeddings.

LEMMA 2.3. *If* F *is a* 2-*manifold with a compact core, then* F *can be standardly embedded in a compact manifold.*

Proof. Since F has a compact core, F can be embedded in a compact manifold H so that int F = int H. We shall, if necessary, alter this embedding so as to satisfy condition (ii). Let $\mathscr{T}$ be the decomposition of H whose nondegenerate elements are those components of $\partial H - \partial F$ that are arcs. Let p be the projection map of H onto the decomposition space $H/\mathscr{T}$. We then have that $H/\mathscr{T}$ is a compact 2-manifold (homeomorphic to H) and $p|F$ is a standard embedding of F in $H/\mathscr{T}$.

Definition. Let M be a 3-manifold and let F be a 2-manifold contained in ∂M. Let U(F) be a subset of M with the following properties:

(i) $F \subseteq U(F)$,

(ii) U(F) is open in M,

(iii) there is a homeomorphism of the pair $(F \times [0, 1), F \times \{0\})$ onto (U(F), F) that maps $\partial F \times [0, 1)$ into ∂M,

(iv) for $0 \leq t < 1$, $F \times [0, t]$ is a closed subset of M.

Then we call U(F) a *proper collar neighborhood* of F.

LEMMA 2.4. (*Improving a compactification.*) *Let* $h: N \to Q$ *be a manifold compactification of a* 3-*manifold* N. *Suppose* $F \subseteq \partial N$ *is a* 2-*manifold such that* F *has a compact core and a proper collar neighborhood* U(F). *Then there exists a new manifold compactification* $h': N \to Q$ *such that each of the following conditions is satisfied:*

(i) *the maps* h *and* h' *are identical on* N - U(F),

(ii) *the closure of* h'(F) *in* Q *is a* 2-*manifold* (*which we denote by* H),

(iii) *the manifold* h'(F) *is standardly embedded in* H.

Definition. If a compactification h' satisfies conditions (ii) and (iii) above, we say h' is *well-behaved* with respect to F.

Proof of Lemma 2.4. We shall obtain h' by composing h with a suitable embedding k of N into N.

By Lemma 2.3, since F has a compact core, there exists a standard embedding of F in a compact manifold $\overline{F}$. By definition of proper collar neighborhood, we may identify U(F) with the subspace $F \times [0, 1)$ of $\overline{F} \times [0, 1)$. Let G be a compact core of F, and define k to be the identity map on the set

$$F \times [1/2, 1) \cup G \times [0, 1) \cup N - U(F).$$

This guarantees that condition (i) will be satisfied.

On the set $(F - \text{int } G) \times [0, 1/2]$, we shall define k to be the composition $k = k_2^{-1} \circ k_3 \circ k_2 \circ k_1$, where k_1, k_2, and k_3 are defined in Steps 1 to 3 below. Each component A of $\overline{F}$ - int G is an annulus. Let B be the closed annulus in $\mathbb{R}^2 \subseteq \mathbb{R}^3$ bounded by the circles of radius 1 and 2 centered at the origin. We identify A with B so that $\partial G \cap A$ is the inner circle. We further identify $A \times [0, 1/2]$ in $\overline{F} \times [0, 1]$ with $B \times [0, 1/2]$ in $\mathbb{R}^2 \times \mathbb{R}^1 = \mathbb{R}^3$. Let p = (0, 0, 2), and let C be the right circular cone whose vertex is p and whose base is the disk in the plane $z = -1/2$ with center $(0, 0, -1/2)$ and radius slightly less than 2.

Step 1. Let k_1 be the homeomorphism of $A \times [0, 1/2]$ onto $A \times [-1/2, 1/2]$ defined by the rule $(s, t) \to (s, 2t - 1/2)$.

Step 2. Let X be the set $A \times [-1/2, 1/2]$ - int C, and let k_2 be the homeomorphism of $A \times [-1/2, 1/2]$ onto X obtained by radial, level-preserving dilation.

Step 3. Let k_3 be the embedding of X into $X \cap \{0 \leq z \leq 1/2\}$ defined as follows: for each straight line L emanating from p, map $L \cap X$ linearly onto $L \cap X \cap \{0 \leq z \leq 1/2\}$ or $L \cap X \cap \{0 < z \leq 1/2\}$ as appropriate.

Since $F \times [0, 1/2]$ is closed in N, k is an embedding of N into N. Since k(int N) = int N, $h'\colon N \to Q$ is a manifold compactification. The closure of k(F) in $\overline{F}$, denoted by H, is compact. Therefore the closure of h'(F) in ∂Q is equal to h(H) and is a compact 2-manifold. Finally, we recall that our original embedding of F in $\overline{F}$ was standard and note that this property was preserved by each of the homeomorphisms used to obtain h'.

Our final lemma is a slight strengthening of Lemma 2.4.

LEMMA 2.4.1. *Let* $h\colon N \to Q$ *be a manifold compactification of a* 3-*manifold* N. *Suppose* $\{F_1, F_2, \cdots\}$ *is a collection of pairwise disjoint* 2-*manifolds in* ∂N *such that each* F_i *has a compact core and a proper collar neighborhood* U_i. *Assume in addition that* $\bigcup_{i=1}^{\infty} F_i$ *is locally connected. Then there exists a manifold compactification* $h'\colon N \to Q$ *that is well-behaved with respect to each* F_i.

Proof. Since each F_i is closed in N and $\bigcup_{i=1}^{\infty} F_i$ is locally connected, we can shrink the neighborhoods U_i one at a time to obtain pairwise disjoint proper collar neighborhoods U_i' of the manifolds F_i. We then apply Lemma 2.4 repeatedly to generate a sequence of embeddings $h_1, h_2, \cdots$ of N into Q such that for each $i > 1$, h_i agrees with h_{i-1} on $N - U_i'$ and h_i is well-behaved with respect to F_i. The sequence (h_i) converges to the desired embedding h'.

COMPACTIFICATION THEOREM 2.5. *Let* M *be a connected* 3-*manifold such that* $\pi_1(M)$ *is finitely generated, and assume that* M *can be expressed as the union of submanifolds* M_i *satisfying the following conditions:*

(i) *each* M_i *is a connected* 3-*manifold possessing a manifold compactification* $h_i\colon M_i \to Q_i$,

(ii) *all but perhaps finitely many* M_i *are* P^2-*irreducible,*

(iii) *for each pair* i, j ($i \neq j$), $M_i \cap M_j$ *is a* 2-*manifold contained in* ∂M_i,

(iv) *for each pair* i, j ($i \neq j$), $M_i \cap M_j$ *has a compact core and has a proper collar neighborhood in* M_i,

(v) *for each pair* i, j ($i \neq j$), $M_i \cap M_j$ *is incompressible in* M_i,

(vi) *for each* i *and each component* F *of* $M_i \cap \bigcup_{j \neq i} M_j$, *there exists* j *such that* F *is a component of* $M_i \cap M_j$,

(vii) *for each* i, $M_i \cap \bigcup_{j \neq i} M_j$ *is locally connected.*

Then M *has a manifold compactification.*

Proof. For each M_i we may, by Lemma 2.4 or 2.4.1, assume $h_i: M_i \to Q_i$ is well-behaved with respect to each intersection $M_i \cap M_j$ $(i \neq j)$. We may also assume that in each M_i, the proper collar neighborhoods of the various surfaces $M_i \cap M_j$ $(i \neq j)$ have pairwise disjoint closures.

We shall first show that any finite union $\bigcup_{i=1}^n M_i$ has a manifold compactification that is well-behaved with respect to each surface $\left(\bigcup_{i=1}^n M_i\right) \cap M_j$ $(j > n)$. For the case $n = 2$, let $F = M_1 \cap M_2$ and let $\overline{F}_i$ $(i = 1, 2)$ be the closure of $h_i(F)$ in Q_i. Since h_i is well-behaved with respect to F, there exists, by Lemma 2.2, an attaching homeomorphism $\bar{g}: \overline{F}_1 \to \overline{F}_2$ whose restriction $g = \bar{g}|h_1(F)$ to $h_1(F)$ is homotopic to $h_2 h_1^{-1}$. Since F has proper collar neighborhoods in M_1 and M_2, the fact that g is homotopic to $h_2 h_1^{-1}$ implies that the manifold $h_1(M_1) \underset{g}{\cup} h_2(M_2)$ is homeomorphic to $M_1 \cup M_2$. Thus the union $Q_1 \underset{\bar{g}}{\cup} Q_2$ is a manifold compactification of $M_1 \cup M_2$ and our embedding $h_1 \vee h_2: M_1 \cup M_2 \to Q_1 \cup Q_2$ is well-behaved with respect to each intersection $(M_1 \cup M_2) \cap M_j$ $(j \neq 1, 2)$. Now let $N = \bigcup_{i=1}^{n-1} M_i$ and suppose that $h: N \to Q$ is a manifold compactification that is well-behaved with respect to each surface $N \cap M_j$ $(j > n)$. By conditions (iii), (iv), (vi), and invariance of domain, the sets $F_i = M_n \cap M_i$ $(i = 1, \cdots, n - 1)$ are separated sets, and so the set $F = M_n \cap N$, which is equal to $\partial M_n \cap \partial N$, is a 2-manifold (with a compact core). Furthermore, we can shrink the proper collar neighborhoods of the F_i to obtain proper collar neighborhoods of F in M_n and in N. We then proceed exactly as in the case $n = 2$ to compactify $N \cup M_n$.

Since $\pi_1(M)$ is finitely generated and almost all the M_i are P^2-irreducible, we can find a finite collection $\{M_1, \cdots, M_n\}$ such that $\bigcup_{i=1}^n M_i$, which we denote by N, is connected, $(\pi_1(N) \to \pi_1(M))$ is surjective, and each M_j $(j > n)$ is P^2-irreducible. Let $h: N \to Q$ be a manifold compactification. We shall show that Q is a manifold compactification of M, by pushing $M - N$ into a collar neighborhood of $\mathrm{fr}(N)$. We do the pushing on one M_j at a time, beginning with one that meets N, and proceeding inductively.

Let j be the smallest index $(j > n)$ for which $M_j \cap N \neq \emptyset$. Since $(\pi_1(N) \to \pi_1(M))$ is surjective and M_j is connected, $M_j \cap N$ is a connected surface, which we denote by F. Since $(\pi_1(N) \to \pi_1(M))$ is surjective and $M_j \cap \bigcup_{k \neq j} M_k$ is incompressible in M, $(\pi_1(F) \to \pi_1(M_j))$ is a surjective isomorphism. Since M_j is P^2-irreducible and F is incompressible, $\pi_2(M_j) = 0$ (Lemma 1.1) and $F \not\approx P^2$ or S^2; thus we also have $\pi_2(F) = 0$. Therefore the inclusion map $F \to M_j$ is a homotopy equivalence. Let $\overline{F}$ be the closure of $h_j(F)$ in Q_j. Then the inclusion $\overline{F} \to Q_j$ is a homotopy equivalence, and by Theorem 3.4 of [1], the pair $(Q_j, \overline{F})$ must then be homeomorphic to $(\overline{F} \times [0, 1], \overline{F} \times \{0\})$.

Since Q_j is just $\overline{F} \times [0, 1]$ and the closures of F in Q_j and Q are homeomorphic, there exists an embedding of $N \cup M_j$ into Q that agrees with h outside a collar neighborhood of F in N. Proceeding inductively, we can define a sequence of manifolds $(\mathcal{M}_k)$ and embeddings $\hat{h}_k: \mathcal{M}_k \to Q$ such that $\bigcup_k \mathcal{M}_k = M$ and $(\hat{h}_k)$ converges to a manifold compactification $h: M \to Q$.

3. APPLICATIONS TO COVERING SPACES

In this section, compact manifolds and submanifolds are polyhedral.

Definition. If a 3-manifold M has the property that for each finitely generated subgroup of $\pi_1(M)$, the covering space $\tilde{M}(H)$ of M corresponding to H has a manifold compactification, then we shall say M has *almost-compact coverings.*

We can restate our original conjecture as follows: *Every compact, connected,* P^2*-irreducible* 3*-manifold has almost-compact coverings.*

Remark. A similar conjecture is presented in [16].

In this section, we apply our compactification theorem to covering spaces. The following theorem is the main tool for this section.

THEOREM 3.1. *Let* M *be a compact, connected* 3*-manifold, and let* H *be a finitely generated subgroup of* $\pi_1(M)$. *Suppose that* M *is the union of connected submanifolds* M_1 *and* M_2 *satisfying the following conditions:*

(i) M_1 *and* M_2 *are compact and* P^2*-irreducible,*

(ii) $M_1 \cap M_2 = \partial M_1 \cap \partial M_2$ *is a* 2*-manifold that is incompressible in each of* M_1, M_2,

(iii) *for each component* F *of* $M_1 \cap M_2$, *the intersection of* $\pi_1(F)$ *with each conjugate of* H *in* $\pi_1(M)$ *is finitely generated,*

(iv) *for each* M_j (j = 1, 2), *the covering space of* M_j *corresponding to a* finitely generated *subgroup of the form* $\pi_1(M_j) \cap g^{-1}Hg$ $(g \in \pi_1(M))$ *has a manifold compactification.*

Then the covering space $\tilde{M}(H)$ *has a manifold compactification.*

Remarks. A proof of Theorem 3.1 is given at the end of this section. It should be noted that condition (iv) is certainly satisfied if each M_j has almost-compact coverings. Condition (iii) is weaker than the assumption that H meets each finitely generated subgroup of $\pi_1(M)$ in a finitely generated group; it follows from conditions (i) and (ii), Lemma 1.3, and Theorem 4.4 of [8], that we could replace condition (iii) with the assumption that for each $g \in \pi_1(M)$, at least one of the groups $gHg^{-1} \cap \pi_1(M_1)$ or $gHg^{-1} \cap \pi_1(M_2)$ is finitely generated.

COROLLARY 3.2. *Let* M *be a* 3*-manifold and suppose that* M *is the union of connected submanifolds* M_1 *and* M_2 *satisfying the following conditions:*

(i) M_1 *and* M_2 *are compact and* P^2*-irreducible,*

(ii) $M_1 \cap M_2 = \partial M_1 \cap \partial M_2$ *is a* 2*-manifold that is incompressible in each of* M_1, M_2,

(iii) *each component of* $M_1 \cap M_2$ *is a disk, annulus, Möbius band, or torus,*

(iv) *each* M_i (i = 1, 2) *has almost-compact coverings.*

Then M *has almost-compact coverings.*

Proof. Let H be a finitely generated subgroup of $\pi_1(M)$. Conditions (i) to (iv) of Theorem 3.1 are clearly satisfied, and therefore $\tilde{M}(H)$ has a manifold compactification.

COROLLARY 3.3. *If* M *is homeomorphic to* $T \times [0, 1]$, $T \times S^1$, *or a twisted line bundle over* T, *where* T *is a connected* 2*-manifold, then* M *has almost-compact coverings.*

Proof. If M is a line bundle over T, then each covering $\tilde{M}(H)$ is a line bundle over the corresponding covering space $\tilde{T}(H)$. If $M \cong T \times S^1$ and T has a compact core, we first construct explicitly the coverings of $S^2 \times S^1$, $P^2 \times S^1$, $D^2 \times S^1$, and $S^1 \times [0, 1] \times S^1$; we can reduce the general problem to these special cases by cutting T along a nonseparating spanning arc or 2-sided simple closed curve, inducting on the rank of $\pi_1(T)$, and invoking Corollary 3.2. The final case we consider is where $\pi_1(T)$ is infinitely generated and $M = T \times S^1$. Let H be a finitely generated subgroup of $\pi_1(M)$. The group $\pi_1(M)$ is the direct sum $\mathbb{Z} \oplus \mathcal{F}$, where $\mathcal{F}$ is a free group with a countably infinite basis $\{a_1, a_2, \cdots\}$. Since H is finitely generated, there is a finite set $a_1, \cdots, a_m$ such that

$$H \subseteq \mathbb{Z} \oplus \langle a_1, \cdots, a_m \rangle \subseteq \pi_1(M).$$

The covering space N of M corresponding to the subgroup $\mathbb{Z} \oplus \langle a_1, \cdots, a_m \rangle$ is the product of S^1 with a (noncompact) surface having a compact core. Since N has almost-compact coverings and $\tilde{M}(H)$ is a covering of N, we conclude that $\tilde{M}(H)$ has a manifold compactification.

COROLLARY 3.4. *A cube-with-handles has almost-compact coverings.*

Definitions. Let T be a solid torus in S^3 and let K be a knot in ∂T. If T is unknotted, we call K a *torus knot.* If T is knotted in the shape of a knot L, we call K a *cable* about L. If K_1 and K_2 are knots that lie on opposite sides of a 2-sphere $S \subseteq S^3$ such that $K_1 \cap S = K_2 \cap S$ is an arc α, then we call the knot $K_1 \cup K_2 - \text{int}\, \alpha$ the *composition* of K_1 and K_2.

COROLLARY 3.5. *If* M *is the closed complement of a regular neighborhood of a torus knot in* S^3, *then* M *has almost-compact coverings.*

Proof. Apply Corollary 3.2, with M_1 and M_2 solid tori and $M_1 \cap M_2$ an annulus.

COROLLARY 3.6. *The class of knots* K *for which a cube with a* K*-knotted hole has almost-compact coverings is closed under the knot operations of composition and cabling.*

Proof. Apply Corollary 3.2, with M_1 a cube with a knotted hole having almost-compact coverings, $M_1 \cap M_2$ an annulus, and M_2 a solid torus (cabling) or a cube with a knotted hole having almost-compact coverings (composition).

Remark. The usefulness of Theorem 3.1 would be greatly enhanced if we could deduce condition (iii) from the hypothesis that H is finitely generated; however, counterexamples exist in a cube with a trefoil-knotted hole ([10], p. 254) and in (surface) $\times S^1$ ([8, Theorem 8.9], which depends on [7]). In view of these examples and Corollaries 3.3 and 3.5, it is possible for a particular manifold M to have some decompositions $M = M_1 \cup M_2$ that are "bad" and others that are "good".

THEOREM 3.7. *Let* M *be a compact, connected,* P^2*-irreducible, sufficiently large* (see Lemma 1.2) 3*-manifold, and let* H *be a finitely generated subgroup of* $\pi_1(M)$. *Assume that the intersection of* H *with each finitely generated subgroup of* $\pi_1(M)$ *is finitely generated. Then* $\tilde{M}(H)$ *has a manifold compactification.*

Proof. By [17, Section 1.6] or [5, Theorem 3], since M is sufficiently large and P^2-irreducible, there exists a sequence of submanifolds $N_1, \cdots, N_n$ of M such that $N_1 = M$, each component of N_n is a 3-cell, and each N_i $(i \geq 2)$ is obtained from N_{i-1} by cutting N_{i-1} open along a properly embedded, connected, incompressible surface F_{i-1}; such a sequence of submanifolds is called a *hierarchy* for M of *length* n. For each compact, connected, P^2-irreducible manifold N possessing a hierarchy, let $\lambda(N)$ denote the length of a shortest hierarchy for N. We shall establish our theorem by induction on the number $\lambda(M)$.

If $\lambda(M) = 1$, then M is a 3-cell and there is nothing to prove. If $\lambda(M) = 2$, then M is a solid torus or solid Klein bottle; in either case, we can explicitly construct and compactify the coverings of M. We now consider the general case.

We wish to express M as a union of submanifolds M_1 and M_2 satisfying conditions (i) to (iv) of Theorem 3.1. Our choices of M_1 and M_2 are made as follows: let $N_1, \cdots, N_{\lambda(M)}$ be a hierarchy for M of minimal length; if N_2 is disconnected, let M_1 and M_2 be the two components of N_2; if N_2 is connected, let M_1 be N_2 and let M_2 be the product neighborhood $F_1 \times [0, 1]$ that we remove from M to create N_2.

Now $M = M_1 \cup M_2$ where M_1 and M_2 are compact, P^2-irreducible manifolds, $\lambda(M_1) < \lambda(M)$, and either $\lambda(M_2) < \lambda(M)$ or M_2 is the product of a surface with an interval. Also, $M_1 \cap M_2$ consists of one or two copies of the incompressible surface F_1. Thus M_1 and M_2 satisfy conditions (i) and (ii) of Theorem 3.1. For each component F of $M_1 \cap M_2$, the intersection of $\pi_1(F)$ with a conjugate of H is isomorphic to the intersection of H with a (finitely generated) conjugate of $\pi_1(F)$. By assumption, such a group is finitely generated, so that M_1 and M_2 satisfy condition (iii). We now wish to verify condition (iv). Let H_j be a finitely generated subgroup of $\pi_1(M_j)$ (j = 1 or 2) of the form

$$H_j = \pi_1(M_j) \cap g^{-1}Hg \qquad \text{for some } g \in \pi_1(M);$$

we need to show that $\tilde{M}_j(H_j)$ has a manifold compactification. If M_j is the product of a surface with an interval, then by Corollary 3.3, M_j has almost-compact coverings. If M_j is not such a product, then $\lambda(M_j) < \lambda(M)$. Thus, by induction, to conclude that $\tilde{M}(H_j)$ has a manifold compactification, it suffices for us to show that the intersection of H_j with any finitely generated subgroup K of $\pi_1(M_j)$ is finitely generated. But

$$K \cap H_j = K \cap \pi_1(M_j) \cap g^{-1}Hg = K \cap g^{-1}Hg \cong gKg^{-1} \cap H ;$$

by assumption on H, $gKg^{-1} \cap H$ is finitely generated.

COROLLARY 3.8. *Let* M *be a compact, connected,* P^2*-irreducible, sufficiently large* 3*-manifold, and let* H *be a finitely generated subgroup of* $\pi_1(M)$ *satisfying either one of the following conditions:*

(i) H *is abelian,*

or

(ii) $H \subseteq \text{image}(\pi_1(\partial M) \to \pi_1(M))$.

Then $\tilde{M}(H)$ *has a manifold compactification.*

Proof. For (i), every subgroup of H is finitely generated. For (ii), by Theorem 8.10 of [8], the intersection of H with each finitely generated subgroup of $\pi_1(M)$ is finitely generated. In either case, the result then follows from Theorem 3.7.

Remark. A new proof of Theorem 8.10 of [8] is given in [9].

COROLLARY 3.9. *Let* M *be a compact, connected,* P^2*-irreducible* 3*-manifold, and let* B *be an incompressible component of* ∂M. *Then* $B \times [0, 1]$ *is a manifold compactification of* $\tilde{M}(\pi_1(B))$.

Proof. Since $\partial M \neq \emptyset$, M is sufficiently large. Therefore, by Corollary 3.8, $\tilde{M}(\pi_1(B))$ has a manifold compactification $Q(\tilde{M})$. Also $\tilde{M}(\pi_1(B))$ has a boundary component $\tilde{B}$ homeomorphic to B, such that the inclusion map of $\tilde{B}$ into $\tilde{M}(\pi_1(B))$ is a homotopy equivalence. By Theorem 3.4 of [1], $Q(\tilde{M})$ is homeomorphic to $\tilde{B} \times [0, 1]$.

Before proving Theorem 3.1, we need to establish several lemmas.

LEMMA 3.10. *Let* $G = A \underset{F}{*} B$ *be a free product of groups* A *and* B *with amalgamated subgroup* F. *If* G *and* F *are finitely generated, then* A *and* B *are finitely generated.*

Proof. This follows from pp. 205-206 of [12].

LEMMA 3.11. *Let* A *be a group with finitely generated isomorphic subgroups* F_0 *and* F_1, *and suppose* $h\colon F_0 \to F_1$ *is an isomorphism. If the group*

$$A * (t) / \{tft^{-1} = h(f) \colon f \in F_0\}$$

is finitely generated, then A *is finitely generated.*

Proof. This is a special case of [10, Lemma 3].

LEMMA 3.12. *Let* N *be a connected submanifold of a connected* 3*-manifold* $\tilde{M}$ *such that* $\pi_1(\tilde{M})$ *is finitely generated. Suppose that each component* F *of* fr(N) *has the following properties:*

(i) F *is a* 2*-manifold in* ∂N,

(ii) F *is incompressible in* $\tilde{M}$,

(iii) $\pi_1(F)$ *is finitely generated,*

(iv) F *is bicollared in* $\tilde{M}$.

Then $\pi_1(N)$ *is finitely generated.*

Proof. Let $\{F_j\}$ be the family of components of fr(N) and let $\{N_i\}$ be the family of closed complementary domains of N in $\tilde{M}$. Since $\pi_1(\tilde{M})$ is finitely generated, there exists a finite collection $N_1, \cdots, N_n$ such that $N \cup N_1 \cup \cdots \cup N_n$ is connected and the inclusion-induced homomorphism

$$i_*\colon \pi_1(N \cup N_1 \cup \cdots \cup N_n) \to \pi_1(\tilde{M})$$

is surjective. Since the surfaces F_j are incompressible in $\tilde{M}$, the homomorphism i_* must be an isomorphism; in particular, $\pi_1(N \cup N_1 \cup \cdots \cup N_n)$ is finitely generated.

Let $N' = N \cup N_1 \cup \cdots \cup N_{n-1}$. We shall show that $\pi_1(N')$ is finitely generated and conclude, by induction on n, that $\pi_1(N)$ is finitely generated. Since $\pi_1(N' \cup N_n)$ is finitely generated, $N' \cap N_n$ has finitely many components $F_1, \cdots, F_m$. If $m = 1$, then Lemma 3.10 applies. If $m \geq 2$, let N" be the manifold obtained by cutting $N' \cup N_n$ open along F_m. Since the surfaces F_j have product neighborhoods, we can view $N' \cup N_n$ as the union of N" and $F_m \times [0, 1]$, joined along the surfaces $F_m \times \{0\}$ and $F_m \times \{1\}$. From Lemma 3.11, we see that $\pi_1(N'')$ must be finitely generated. We now cut N" open along F_{m-1} and repeat the above argument; after $(m - 1)$ applications of Lemma 3.11 and one application of Lemma 3.10, we see that $\pi_1(N')$ is finitely generated.

Proof of Theorem 3.1. Let $p: \tilde{M}(H) \to M$ be the covering map and let $\{N_i\}$ be the set of components of $p^{-1}(M_1)$ and $p^{-1}(M_2)$. We shall show that the manifolds N_i satisfy conditions (i) to (vii) of Theorem 2.5 to conclude that $\tilde{M}(H)$ has a manifold compactification.

Condition (i). Each N_i is the covering space of M_j ($j = 1$ or 2) corresponding to a subgroup of $\pi_1(M_j)$ of the form $\pi_1(M_j) \cap g^{-1} H g$ ($g \in \pi_1(M)$). Since each component F of $M_1 \cap M_2$ has the property that the intersection of $\pi_1(F)$ with each conjugate of H in $\pi_1(M)$ is finitely generated, the components of $p^{-1}(F)$ have finitely generated fundamental groups. Thus, by Lemma 3.12, $\pi_1(N_i)$ is finitely generated. Hypothesis (iv) of Theorem 3.1 then guarantees that N_i has a manifold compactification.

Condition (ii). Since M_1 and M_2 are P^2-irreducible, it follows from Lemma 1.3 that each N_i is P^2-irreducible.

Conditions (iii) and (vii). These follow from the fact that p is a local homeomorphism.

Condition (iv). Since $\pi_1(\tilde{M}(H))$ is finitely generated, each intersection $N_i \cap N_k$ ($i \neq k$) has at most a finite number of components. As noted above, the fundamental group of each such component is finitely generated. By lifting a bicollar neighborhood of $M_1 \cap M_2$, we obtain the desired proper collar neighborhoods.

Condition (v). Since $M_1 \cap M_2$ is incompressible, each component of $p^{-1}(M_1 \cap M_2)$ is incompressible.

Condition (vi). Each component $\tilde{F}$ of $N_i \cap \bigcup_{j \neq i} N_j$ is a component of $p^{-1}(M_1 \cap M_2)$. For whichever $j \neq i$ we have $\tilde{F} \cap N_j \neq \emptyset$, $\tilde{F}$ is a component of $N_i \cap N_j$.

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