UNIVALENCE AND BOUNDED MEAN OSCILLATION

Albert Baernstein II

1. INTRODUCTION

We shall denote by D the unit disk $|z| < 1$ and by T the boundary of D. A function $g \in L^1(T)$ is said to be of *bounded mean oscillation*, $g \in$ BMO, if there exists a constant $C = C(g)$ such that

$$\frac{1}{|I|}\int_I |g(e^{i\theta}) - g_I|\, d\theta \leq C$$

for every interval (circular arc) $I \subset T$. Here, $|I|$ denotes the (one-dimensional) Lebesgue measure of I and g_I is the average value of g over I,

$$g_I = \frac{1}{|I|}\int_I g(e^{i\theta})\, d\theta\, .$$

The class BMO was introduced by John and Nirenberg [10]. They showed that a BMO function g, which is *a priori* assumed only to be in L^1, in fact satisfies the much stronger integrability condition $e^{\alpha|g|} \in L^1(T)$ for some positive α.

BMO functions have attracted considerable attention in recent years since the discovery by Fefferman ([5], [6]) that they play a very important role in certain aspects of harmonic analysis. Among other things, Fefferman proved [6, Theorem 3] that a real valued function $u \in L^1(T)$ is in BMO if and only if u has the form

(1) $$u = u_1 + \tilde{u}_2, \qquad \text{where } u_1, u_2 \in L^\infty(T).$$

Here, $\tilde{u}_2$ denotes the conjugate function of u_2. In particular, BMO properly contains $L^\infty(T)$, while the John-Nirenberg result mentioned above shows that $L^p(T) \supset$ BMO for all $p < \infty$.

We remark that the authors cited so far actually studied BMO functions defined on R^n, but all of their results are still valid, and slightly easier to prove, for functions defined on T.

Suppose now that f is an analytic univalent function in D. It is well known (see, *e.g.*, [4]) that $f \in H^p$ (the usual Hardy class) for $0 < p < 1/2$. Thus the radial limits $f(e^{i\theta})$ of f exist for almost all θ, and it is easy to show that

$$\log|f(e^{i\theta})| \in L^p(T)$$

for all $p < \infty$, although $\log|f(e^{i\theta})|$ need not be in $L^\infty(T)$. Since BMO is situated between L^p and L^∞, it is natural to ask whether $\log|f(e^{i\theta})| \in$ BMO. This question was apparently first considered by Cima and Petersen [2], who showed that the

Received June 16, 1976.
This research was supported in part by the National Science Foundation.

Michigan Math. J. 23 (1976).

answer is affirmative for certain subclasses of univalent functions. In this paper, I shall prove the following more precise result, which implies that the answer is always affirmative.

THEOREM 1. *Suppose that* f *is analytic and univalent in* D. *Then, for each* $p \in (0, 1/2)$, *there exist functions* $u_1, u_2 \in L^\infty(T)$ *such that* $\|u_2\|_\infty < \pi/2p$ *and, for almost all real* θ,

$$\log|f(e^{i\theta})| = u_1(e^{i\theta}) + \tilde{u}_2(e^{i\theta}).$$

In view of Fefferman's result (1), Theorem 1 certainly implies that $\log|f(e^{i\theta})| \in \mathrm{BMO}$. Moreover, (1) shows that the linear space BMO is closed under conjugation. Thus $u \in \mathrm{BMO} \Leftrightarrow u + i\tilde{u} \in \mathrm{BMO}$, and we obtain immediately from Theorem 1

THEOREM 2. *Suppose that* f *is analytic and univalent in* D. *Then*

(a) $$\log f(e^{i\theta}) \in \mathrm{BMO} \quad \textit{if } f \textit{ has no zero in } D;$$

(b) $$\log \frac{f(e^{i\theta})}{e^{i\theta} - a} \in \mathrm{BMO} \quad \textit{if } f \textit{ has a zero at } z = a \in D.$$

Part (b) is a consequence of the fact that

$$\log\left[\frac{1 - \bar{a}z}{z - a} f(z)\right]$$

is analytic and single-valued in D, and the real part of its boundary function is $\log|f(e^{i\theta})|$.

Two especially interesting subclasses of the set of univalent functions are the classes S and S_0 of functions f satisfying respectively

(S) $$f(0) = 0, \qquad f'(0) = 1;$$

(S_0) $$f(0) = 1, \qquad f \text{ has no zero in } D.$$

The results of [1] provide one with very good control over the modulus $|f|$ for $f \in S$ or $f \in S_0$. For example, if $f \in S$, then Theorems 1 and 2 of [1] assert that

$$\int_0^{2\pi} \Phi(\pm\log|f(re^{i\theta})|)\, d\theta \le \int_0^{2\pi} \Phi(\pm\log|k(re^{i\theta})|)\, d\theta \tag{2}$$

for every convex increasing function Φ, where $k(z) = z/(1 - z)^2$ is the Koebe function. It is easy to check directly that $\log|k(e^{i\theta})| \in \mathrm{BMO}$. However, this together with (2) does not imply $\log|f(e^{i\theta})| \in \mathrm{BMO}$. The class BMO is more subtle than the L^p classes, and the results of the present paper yield interesting information about the *argument* $\arg(f(e^{i\theta})/e^{i\theta})$ for $f \in S$, which does not seem directly accessible by the methods of [1]. There is a possibility that the present results, combined with those of [1], could be put to some use in the study of coefficient problems for functions in S.

2. PROOF OF THEOREM 1

The proof of Theorem 1 is based on deep results of Hunt, Muckenhoupt, and Wheeden [9], whose main concern was to characterize those weight functions u on T for which the conjugate function operator is bounded on $L^p(T, u\,d\theta)$. Let u be a nonnegative function on T such that both $u \in L^1(T)$ and $1/u \in L^1(T)$. Denote by P(u, z) the Poisson integral of u,

$$P(u, z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\,\Re\left[\frac{1+ze^{-i\phi}}{1-ze^{-i\phi}}\right]d\phi, \quad z \in D.$$

Theorem 2 of [9] then asserts that the following two conditions are equivalent.

(3) $$u = \exp(u_1 + \tilde{u}_2), \quad \text{with } u_1, u_2 \in L^\infty \text{ and } \|u_2\|_\infty < \pi/2.$$

(4) There exists a constant $C = C(u)$ such that $P(u, z)\,P(1/u, z) \leq C$, for all $z \in D$.

We shall establish Theorem 1 by showing that, for each $p \in (0, 1/2)$,

(5) $$P(|f|^p, z)\,P(|f|^{-p}, z) \leq C, \quad z \in D,$$

for every univalent function f. The constant C will depend on p, but our proof shows that it may be chosen to be independent of f.

Suppose first that f has no zero in D. We may assume that $f(0) = 1$, so that $f \in S_0$. Take $p \in (0, 1/2)$ and set $h(\zeta) = P(|f|^p, \zeta)$. Fix $z \in D$ and define

$$Q(\zeta) = \frac{\zeta + z}{1 + \bar{z}\zeta}, \quad F(\zeta) = f(Q(\zeta)).$$

Then $h(Q(\zeta)) \equiv P(|F|^p, \zeta)$. This may be established by noting that both functions are harmonic in D with nontangential boundary values $|f(Q(e^{i\theta}))|^p$ a.e. on T, and by then showing, using well known facts which may be found in [4], that $h \circ Q$ is the Poisson integral of its boundary function. Alternatively, the identity can be checked by direct computation, after changing variables in one of the defining integrals. In particular, we have

(6) $$P(|f|^p, z) = h(z) = h(Q(0)) = P(|F|^p, 0) = \frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p\,d\theta.$$

Since F is univalent and zero-free, Theorem 6 of [1] implies that

(7) $$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p\,d\theta \leq C_p\,|F(0)|^p,$$

where $C_p = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1+e^{i\theta}}{1-e^{i\theta}}\right|^{2p}d\theta$. Since $F(0) = f(z)$, (6) and (7) yield

(8) $$P(|f|^p, z) \leq C_p\,|f(z)|^p.$$

But $f \in S_0$ if and only if $1/f \in S_0$. Hence, (8) holds with $1/f$ in place of f, and we deduce

$$P(|f|^p, z)\, P(|f|^{-p}, z) \leq C_p^2$$

for all $z \in D$. Thus, (5) holds if $f \in S_0$.

Assume next that f is univalent in D and that $f(0) = 0$. We may assume for the proof of (5) that $f'(0) = 1$, so that $f \in S$. Take $p \in (0, 1/2)$, $z \in D$, and define again $Q(\zeta) = (\zeta + z)/(1 + \bar{z}\zeta)$. As before (*cf.* (6)) we have

$$P(|f|^p, z) = \frac{1}{2\pi}\int_0^{2\pi} |f(Q(e^{i\theta}))|^p \, d\theta. \tag{9}$$

To estimate the integral on the right, define, for $z \neq 0$,

$$\beta = \frac{f'(z)(1 - |z|^2)}{f(z)}, \qquad g(\zeta) = \frac{1}{\beta}\,\frac{f(Q(\zeta)) - f(z)}{f(z)}.$$

Then $g \in S$, and (2) gives

$$\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta \leq B_p, \qquad \text{where } B_p = \frac{1}{2\pi}\int_0^{2\pi} |k(e^{i\theta})|^p \, d\theta.$$

Since $f \circ Q = f(z)(\beta g + 1)$, we deduce

$$\frac{1}{2\pi}\int_0^{2\pi} |f(Q(e^{i\theta}))|^p \, d\theta \leq |f(z)|^p (|\beta|^p B_p + 1).$$

Combined with (9), this yields

$$P(|f|^p, z) \leq |f(z)|^p (|\beta|^p B_p + 1). \tag{10}$$

The distortion theorem for f'/f [7, p. 4] shows that $|\beta| \leq 4|z|^{-1}$. Hence, if $|z| \geq 1/2$, then it follows from (10) that

$$P(|f|^p, z) \leq |f(z)|^p (8^p B_p + 1), \qquad |z| \geq 1/2. \tag{11}$$

On the other hand, if $|z| \leq 1/2$, then

$$P(|f|^p, z) \leq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p \frac{1 + |z|}{1 - |z|} \, d\theta \leq 3B_p. \tag{12}$$

Our aim is to show that $P(|f|^p, z)\, P(|f|^{-p}, z) \leq C$, independently of z. Since $|f(e^{i\theta})| \geq 1/4$, by the Koebe one-quarter theorem, we have $P(|f|^{-p}, z) \leq 4^p$ for all z. Thus, in view of (11) and (12), the desired result will follow from an estimate of the form

$$P(|f|^{-p}, z) \leq A\, |f(z)|^{-p}, \qquad |z| \geq 1/2. \tag{13}$$

To prove (13), we shall make use of an extension of (2) proved recently by Kirwan and Schober [11]. For $0 < m < 1$, let $S(m)$ be the set of all functions F meromorphic and univalent in D with $F(0) = 0$, $F'(0) = 1$, $F(m) = \infty$. Let

$$k_m(z) = \frac{mz}{(m - z)(1 - mz)} .$$

Kirwan and Schober proved [11, Theorem 2] that, for $F \in S(m)$,

$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |k_m(e^{i\theta})|^p d\theta .$$

A simple computation, which we leave to the reader, shows that

$$|k_m(e^{i\theta})| \leq |k(e^{i\theta})|,$$

and hence

$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p d\theta \leq B_p. \tag{14}$$

Returning now to our function $f \in S$, we have, with the same Q as in (9),

$$P(|f|^{-p}, z) = \frac{1}{2\pi}\int_0^{2\pi} |f(Q(e^{i\theta}))|^{-p} d\theta . \tag{15}$$

Define, with β as before,

$$F(\zeta) = \frac{1}{\beta}\left[1 - \frac{f(z)}{f(Q(\zeta))} \right].$$

Then F is univalent in D, $F(0) = 0$, $F'(0) = 1$, and $F(-z) = \infty$. Thus

$$e^{i\alpha} F(e^{-i\alpha}\zeta) \in S(|z|)$$

for a suitable real number α, and hence (14) holds for F. Now

$$\frac{1}{f \circ Q} = \frac{1}{f(z)}(1 - \beta F),$$

and so, by (15),

$$P(|f|^{-p}, z) = |f(z)|^{-p}\frac{1}{2\pi}\int_0^{2\pi} |1 - \beta F(e^{i\theta})|^p d\theta \leq |f(z)|^{-p}(1 + |\beta|^p B_p).$$

Since $|\beta| \leq 8$ for $|z| \geq 1/2$, we have proved (13), which completes the proof of (5) when f has a zero at the origin.

Finally, suppose that f is univalent in D and that $f(a) = 0$ for some $a \in D$. Let $Q(z) = (z + a)/(1 + \bar{a}z)$. Then $f \circ Q$ has a zero at $z = 0$ and so, by the case already proved,

(16) $$P(|f \circ Q|^p, z)\, P(|f \circ Q|^{-p}, z) \le C \quad (0 < p < 1/2),$$

where C depends only on p. But, as we have seen,

$$P(|f \circ Q|^p, z) = P(|f|^p, Q(z)),$$

and a similar equation holds with -p in place of p. It follows that (16) holds with f in place of $f \circ Q$, and the proof of Theorem 1 is complete.

3. CONCLUDING REMARKS

The Hunt-Muckenhoupt-Wheeden result (3) $\Leftrightarrow$ (4), upon which our proof of Theorem 1 is based, is deduced via a rather complicated argument involving Calderón-Zygmund decompositions, the Marcinkiewicz interpolation theorem, and an earlier nonconstructive existence theorem of Helson and Szegö [8]. Thus, our deduction that $\log|f(e^{i\theta})| \in$ BMO is apparently quite roundabout, using as it does not only (3) $\Leftrightarrow$ (4) but also Fefferman's theorem. However, from the inequality $P(|f|^p, z)\, P(|f|^{-p}, z) \le C$ proved in the present paper, it is in fact easy to show directly that $\log|f(e^{i\theta})| \in$ BMO. This may be accomplished by proving (c) $\Rightarrow$ (a) of Theorem 2, and also Lemma 5 of [9], both of which are elementary. A different proof that $\log|f(e^{i\theta})| \in$ BMO has recently been found by Cima and Schober [3].

If, in Theorem 1, we form the Poisson integrals of u_1 and u_2 and complete them to analytic functions, we are led to a factorization theorem for univalent functions. For zero-free functions it may be stated as follows.

THEOREM 3. *If* $f \in S_0$, *then for each* $p \in (0, 1/2)$ *there exist functions* B *and* F *analytic in* D *such that*

$$f(z) = B(z)[F(z)]^{1/p}, \quad z \in D,$$

where $B \in H^\infty$, $1/B \in H^\infty$, *and* $\Re F(z) > 0$.

The functions B and F above depend on p, and require $p < 1/2$. It would be very interesting if we could pass to the limit $p = 1/2$, and thereby factor $f \in S_0$ into a bounded function times one which is subordinate to a conformal map onto $\mathbb{C} - (-\infty, 0)$.

It is a pleasure to acknowledge the stimulating conversations I have had on the subject of this paper with J. A. Cima, W. E. Kirwan, and G. Schober.

REFERENCES

1. A. Baernstein II, *Integral means, univalent functions and circular symmetrization.* Acta Math. 133 (1974), 139-169.

2. J. A. Cima and K. E. Petersen, *Some analytic functions whose boundary values have bounded mean oscillation.* Math. Z. 147 (1976), 237-247.

3. J. A. Cima and G. Schober, BMOA *and logarithms of* H^p *functions.* Math. Z., to appear.

4. P. L. Duren, *The Theory of* H^p *Spaces.* Academic Press, New York, 1970.

5. C. Fefferman, *Characterizations of bounded mean oscillation.* Bull. Amer. Math. Soc. 77 (1971), 587-588.

6. C. Fefferman and E. M. Stein, H^p *spaces of several variables.* Acta Math. 129 (1972), 137-193.

7. W. K. Hayman, *Multivalent Functions.* Cambridge University Press, 1958.

8. H. Helson and G. Szegö, *A problem in prediction theory.* Ann. Mat. Pura Appl. (4) 51 (1960), 107-138.

9. R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform.* Trans. Amer. Math. Soc. 176 (1973), 227-251.

10. F. John and L. Nirenberg, *On functions of bounded mean oscillation.* Comm. Pure Appl. Math. 14 (1961), 415-426.

11. W. E. Kirwan and G. Schober, *Extremal problems for meromorphic univalent functions.* J. Analyse Math., to appear.

Department of Mathematics
Washington University
St. Louis, Missouri 63130