REMARKS ON FLAT MODULES

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0. INTRODUCTION

The property of a commutative ring A that finitely generated flat A-modules be projective is known to be a property of the topology of Spec(A), the set of prime ideals of A [10]. Similarly, the property that pure ideals be generated by idempotents depends only on the topology of Spec(A) [8]. In fact these properties do not depend on the Zariski topology as much as on a weaker topology on Spec(A). In recent years several results have been obtained about the stability of these properties. The purpose of this note is to determine the extent to which these results follow from topological properties.

In Section 1, we show that the above properties are equivalent to very simple conditions on a certain quotient space of Spec(A). In Section 2, we study the extent to which the above properties are inherited from an A-algebra, or by an A-algebra, in terms of these quotient spaces. It becomes apparent that more than just the map induced on the quotient spaces by the structure homomorphism $A \to B$ is involved in this stability. In Section 3, we show that the fact that a polynomial ring or power series ring over A has either of these properties if and only if A does, is due to the fact that the relevant topological spaces are homeomorphic. We conclude with some remarks about the relationship between the property that finitely generated flat A-modules be projective and the property that finite type flat A-algebras be finitely presented.

In this note all rings are commutative with identity. If M is an A-module and $p \in \mathrm{Spec}(A)$, then $\mathrm{rk}_M(p)$ denotes the dimension of $M \otimes_A k(p)$ as a vector space over $k(p) = R_p/pR_p$. We will label the conditions we are interested in as in [18] and [19].

(F) Every pure ideal of A is generated by an idempotent.

(f) Every pure ideal of A is generated by idempotents.

A ring A satisfies (F) if and only if every finitely generated flat A-module is projective. The papers [3], [10], [16], [17], and [18] are good references for F-rings. See [8] and [19] for f-rings.

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1. TOPOLOGICAL CRITERIA

We consider the C- and D-relations of Lazard on a topological space X. The *D-relation* on X is the equivalence relation generated by the relation $x \in \overline{\{y\}}$ $(x, y \in X)$, where $\overline{W}$ denotes the closure of a subset W of X. We put the quotient topology on the set X/D of D-equivalence classes of X. The weak topology on X

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induced by the canonical map $\mu: X \to X/D$ is called the D-*topology* on X. We call a subset W of X D-*open* or D-*closed* if it is open or closed in the D-topology.

The C-*relation* on X is obtained by taking the equivalence class of a point $x \in X$ to be the intersection of the D-open sets and the D-closed sets of X containing x. We give X/C the quotient topology obtained from the canonical map $\lambda: X \to X/C$. The quotient topology on X/C induced from the original topology on X is the same as the quotient topology on X/C induced by the D-topology on X. Also, since the D-closed sets and D-open sets of X are stable under the C-relation, λ is an open and closed map when X is given the D-topology. Thus, the weak topology on X induced by λ is just the D-topology; that is, we need not consider a C-topology on X.

If $u: Y \to X$ is a continuous map, then we get a commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{u} & X \\ {\scriptstyle \mu(Y)}\downarrow & & \downarrow{\scriptstyle \mu(X)} \\ Y/D & \xrightarrow{u_D} & X/D \\ {\scriptstyle \tau(Y)}\downarrow & & \downarrow{\scriptstyle \tau(X)} \\ Y/C & \xrightarrow{u_C} & X/C \end{array} .$$

Further, if X and Y are given the D-topologies, then u is continuous, and μ, τ, and $\lambda = \tau \circ \mu$ are open and closed as well as continuous. In fact, if we let $\mathscr{D}(X)$ denote the space X with the D-topology, we get the following commutative diagram of natural transformations of functors on the category of topological spaces and continuous maps:

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{D}(X) \\ \downarrow & \swarrow & \downarrow \\ X/D & \longrightarrow & X/C \end{array} .$$

The use of the C- and D-relations in the study of flat and projective modules goes back at least to H. Bass [1] and D. Lazard [10]. Lazard showed that if $p, q \in X = \mathrm{Spec}(R)$, then p and q are C-related if and only if $rk_M(p) = rk_M(q)$ for every finitely generated flat R-module M if and only if $rk_M(p) = rk_M(q)$ for every projective R-module M. He also showed that the property (F) is a property of the space X with the D-topology. S. Jøndrup [8] showed that (f) is also a property of this space. Thus, one is led to consider the space X/C, where the properties (F) and (f) become very simple.

For the rest of this section, A is a ring and $X = \mathrm{Spec}(A)$.

THEOREM 1.1. *The following properties of* A *are equivalent.*

(1) A *satisfies* (F).

(2) *The* D-*open sets of* X *are closed.*

(3) X/C *is discrete, thus finite.*

Proof. (1) $\Leftrightarrow$ (2) [10, Theorem 5.7].

(2) $\Rightarrow$ (3) Since X is compact, (2) implies that X with the D-topology is a finite disjoint union of trivial spaces which are the C-equivalence classes of X. Thus X/C is finite and discrete.

(3) $\Rightarrow$ (1) Since for every D-open or D-closed set W of X, $\lambda^{-1}\lambda(W) = W$, (3) $\Rightarrow$ (1).

THEOREM 1.2. *The following properties of* A *are equivalent.*

(1) A *satisfies* (f).

(2) *Any* D-*closed subset of* X *is an intersection of open-closed subsets of* X.

(3) X/C *has a basis of open-closed sets.*

(4) X/C *is totally disconnected.*

Proof. (1) $\Leftrightarrow$ (2) [8, Theorem 3.3].

(4) $\Rightarrow$ (3) This follows since X/C is compact [11, p. 46, Proposition 8.6].

The other implications follow immediately since λ: X $\to$ X/C is open and closed when X is given the D-topology.

2. STABILITY

For the rest of this note, we let ϕ: A $\to$ B be a homomorphism of commutative rings. We denote the map Spec(ϕ): Spec(B) $\to$ Spec(A) by u: Y $\to$ X and let u_D: Y/D $\to$ X/D and u_C: Y/C $\to$ X/C be the induced maps. We use λ for both the map X $\to$ X/C and the map Y $\to$ Y/C when no confusion can arise. We do the same for μ and τ. If I is an ideal of a ring R, then V(I) denotes the closed set $\{p \in \mathrm{Spec}(R): I \subset p\}$ of Spec(R). The D-closed subsets of Spec(R) are of the form V(I) where I is a pure ideal of R, and the map I $\mapsto$ V(I) gives a one-to-one correspondence between the pure ideals of R and the D-closed subsets of Spec(R) [3, Corollary 3.6]. If I is a principal ideal of R, we call the closed set V(I) a *principal closed set* and its complement a *principal open set.*

We will need the following result.

LEMMA 2.1. *Let* I *be an ideal of* B. *Then* $\mu(V(I \cap A)) = \mu \circ u(V(I))$.

Proof. Clearly $u(V(I)) \subset V(I \cap A)$, so $\mu \circ u(V(I)) \subset \mu(V(I \cap A))$. Now let $z \in \mu(V(I \cap A))$. Let $p \in V(I \cap A)$ be such that $\mu(p) = z$. Then p contains a prime ideal p' which is minimal over $I \cap A$. Since $A/(A \cap I) \to B/I$ is injective, there exists $q \in V(I)$ such that $u(q) = p'$ [9, p. 41, Exercise 1]. Then

$$\mu \circ u(q) = \mu(p') = \mu(p) = z,$$

and hence $z \in \mu \circ u(V(I))$.

Remark. It follows also that $\lambda V(I \cap A) = \lambda u(V(I))$.

PROPOSITION 2.2. *If* A *satisfies* (f), *then* u_C *is a closed map.*

Proof. This follows since Y/C is compact and X/C is a Hausdorff space.

COROLLARY 2.3. *If* B *is an* F-*ring and* u_C *is onto, then the following are equivalent.*

(1) u_C *is closed.*

(2) A *is an* F-*ring.*

(3) A *is an* f-*ring.*

(4) X/C *is a Hausdorff space.*

Proof. This is immediate from Theorems 1.1 and 1.2.

It is well known that if either $B \supset A$ or $B = A/J$, where J lies in the Jacobson radical of A, and if M is a finitely generated flat A-module such that $B \otimes_A M$ is B-projective, then M is A-projective. In particular, if B satisfies (F), then so does A. A topological result which includes these theorems would require conditions on u which imply that if Y/C is discrete then X/C is discrete, or more generally, if $W \subseteq X/C$ is closed and $u_C^{-1}(W)$ is open, then W is open. To get substantial information about X/C from Y/C, one should have u_C onto, as is the case of either $A \subset B$ or $B = A/J$, where J is contained in the Jacobson radical of A. A sufficient condition on u is that u_C be closed and onto. But this does not always hold for $A \subset B$, since u_C closed and onto would imply that the condition (f) passes from B to the subring A, and this is false [8]. A difficulty is that since $V(\ker \phi)$ is not necessarily D-closed, the C-relation on $V(\ker \phi) = \mathrm{Spec}(A/\ker \phi)$ is not in general the same as the C-relation on $V(\ker \phi)$ induced by the C-relation on X. What is needed is a slightly stronger condition on u than u_C being onto.

LEMMA 2.4. *If* $\phi: A \to B$, $I \subset A$ *is pure, and* $IB = eB$, *where* e *is an idempotent, then* $e \in \phi(I)$. *Thus,* $\phi(I) = e\phi(A)$.

Proof. Let $e = \sum_{i=1}^n a_i b_i$ ($a_i \in I$, $b_i \in B$). Then since I is a pure ideal of A, there exists $e_0 \in I$ such that $e_0 a_i = a_i$, $i = 1, 2, \cdots, n$ [2, Chapter 1, Section 2, Exercise 23]. Then

$$e = e_0 \sum a_i b_i = e_0 e.$$

Further, $\phi(e_0) \in IB$ implies $\phi(e_0) = \phi(e_0)e = e_0 e = e$. Thus, $e \in \phi(I)$.

To show $\phi(I) = e\phi(A)$, let $a \in I$. Then $\phi(a) = e\phi(a)$, since e is an idempotent and $IB = eB$.

It follows from the above lemma that if B satisfies (F) and $\ker \phi$ lies in the Jacobson radical of A, then A satisfies (F). The next three results are generalizations of this.

THEOREM 2.5. *If for every principal Zariski closed subset* W *of* X *such that* $u^{-1}(W)$ *is open and closed, and for every* $p \in W$, *there exists* $p_1 \in W \cap u(Y)$ *such that* $\lambda(p) = \lambda(p_1)$, *and if* $H \subset X$ *is* D-*closed and* $u^{-1}(H)$ *is open, then* H *is open.*

Proof. Let $I \subseteq A$ be pure. Then $IB = e_0 B$ for some $e_0 \in I$ with $\phi(e_0)$ idempotent. To show $e_0 A = I$, it suffices to show $V(e_0 A) \subseteq V(I)$. Let $p \in V(e_0 A)$. Since $u^{-1}V(e_0 A) = V(e_0 B)$ is open-closed, there exists $p_1 \in V(e_0 A) \cap u(Y)$ such that $\lambda(p) = \lambda(p_1)$. Then

$$p_1 \in uu^{-1}(V(e_0 A)) = u(V(e_0 B)) = uV(IB) = uu^{-1}V(I) \subseteq V(I).$$

But $\lambda(p) = \lambda(p_1)$ and $p_1 \in V(I)$ imply $p \in V(I)$.

COROLLARY 2.6. *If for every principal Zariski closed subset* W *of* X *such that* $W \cap V(\ker \phi)$ *is open-closed in* $V(\ker \phi)$, *and for every* $p \in W$, *there exists* $p_1 \in W \cap u(Y)$ *such that* $\lambda(p) = \lambda(p_1)$, *and if* I *is a pure ideal of* A *and* IB *is* B-*projective, then* I *is* A-*projective.*

Proof. If $W \cap V(\ker \phi)$ is open-closed, then $u^{-1}(W) = u^{-1}(W \cap V(\ker \phi))$ is open-closed.

COROLLARY 2.7. *If* $Y/C \to \operatorname{Spec}(A/aA)/C$ *is onto for every* $a \in \ker \phi$, *if* I *is a pure ideal of* A, *and if* IB *is* B*-projective, then* I *is projective.*

Proof. By Lemma 2.4, $IB = e_0 B$, where $e_0 \in I$ and $\phi(e_0)$ is idempotent. Thus $a = e_0 - e_0^2 \in \ker \phi$. To show $V(e_0 A) \subseteq V(I)$, let $p \in V(e_0 A)$. By hypothesis, there exists $p_1 \in V(a) \cap u(Y)$ with p_1 C-related to p in $\operatorname{Spec}(A/aA)$. But $V(e_0 A) \cap V(aA)$ is open and closed in $V(a)$ and $p \in V(e_0 A)$, so $p_1 \in V(e_0 A)$. Thus

$$p_1 \in uu^{-1} V(e_0 A) = uu^{-1} V(I) \subseteq V(I).$$

Since p_1 is C-related to p in $V(aA)$, p_1 is C-related to p in X. Thus $p \in V(I)$.

It is clear that the conclusions of Theorem 2.5 and Corollaries 2.6 and 2.7 imply that if L is a finitely generated flat A-module such that $B \otimes_A L$ is B-projective, then L is A-projective.

The question of when condition (F) is inherited by B reduces to the question of when X/C discrete implies Y/C discrete, and similarly for condition (f). Since u_C is closed whenever A satisfies (f), the question reduces to when u_C is one-to-one. If A satisfies (F) and each pure ideal I of B is of the form $I_0 B$, where I_0 is a pure ideal of A, then clearly B satisfies (F), and similarly for (f). If u_C is onto, then this is exactly the condition that u_C be one-to-one (and thus a homeomorphism).

THEOREM 2.8. *If* $u_C\colon Y/C \to X/C$ *is a closed surjective map, then* u_C *is a homeomorphism if and only if every pure ideal* I *of* B *is of the form* $I_0 B$ *for some pure ideal* I_0 *of* A.

Proof. ($\Leftarrow$) Let $q_1, q_2 \in Y$ with $\lambda u(q_1) = \lambda u(q_2)$. Let $V(I)$ be a D-closed set containing q_1, with $I \subset B$ pure. Then $I = I_0 B$, where $I_0 \subset A$ is a pure ideal, and $uV(I) \subset V(I_0)$. But $V(I_0)$ is D-closed and contains $u(q_1)$, so $u(q_2) \in V(I_0)$. Thus $q_2 \in u^{-1} V(I_0) = V(I_0 B) = V(I)$. Hence, every D-closed set containing q_1 contains q_2. The same holds for D-open sets by reversing the roles of q_1 and q_2. Thus $\lambda(q_1) = \lambda(q_2)$.

($\Rightarrow$) Let I be a pure ideal of B. Then since u_C is closed, $u_C \lambda V(I) = \lambda u(V(I))$ is closed. Thus $\lambda^{-1} \lambda u(V(I))$ is a D-closed subset of X, and so $\lambda^{-1} \lambda u(V(I)) = V(I_0)$, where I_0 is a pure ideal of A. Since I and $I_0 B$ are pure, to show $I = I_0 B$, it suffices to show $V(I) = V(I_0 B)$, or $\lambda V(I) = \lambda V(I_0 B)$, since these sets are C-saturated. But since u_C is a homeomorphism, it suffices to show that $u_C \lambda V(I) = u_C \lambda V(I_0 B)$; that is, $\lambda u V(I) = \lambda u V(I_0 B)$. This holds if and only if $\lambda^{-1} \lambda u(V(I)) = \lambda^{-1} \lambda u(V(I_0 B))$. By the definition of I_0, $\lambda^{-1} \lambda u(V(I)) = V(I_0)$. Thus we need only show that $V(I_0) = \lambda^{-1} \lambda u(V(I_0 B))$. Since

$$V(I_0) = \lambda^{-1} \lambda(V(I_0)) \quad \text{and} \quad \lambda^{-1} \lambda u(V(I_0 B)) = \lambda^{-1} \lambda(V(I_0 B \cap A)),$$

it is clear that $\lambda^{-1} \lambda u(V(I_0 B)) \subset V(I_0)$. Let $p \in V(I_0)$. Since u_C is onto, there exists $q \in Y$ such that $u_C \lambda(q) = \lambda(p)$. Thus $\lambda u(q) = \lambda(p)$, and since $V(I_0)$ is C-saturated, $u(q) \in V(I_0) \Rightarrow q \in V(I_0 B) \Rightarrow p \in \lambda^{-1} \lambda u(V(I_0 B))$.

Remarks. The "if" part of the above proof did not use the assumption that u_C is onto. A condition related to the condition in the above theorem is that for every pure ideal I of B, the radical of $I \cap A$ coincides with the radical of I_0 for some pure ideal I_0 of A. This condition is implied by u being closed and implies u_D and u_C are closed maps.

3. POLYNOMIAL AND POWER SERIES RINGS

It is known that a ring A has property (F) if and only if a polynomial or power series ring over A has this property [3], [17], [18]. Similarly, a ring A satisfies (f) if and only if a polynomial ring satisfies (f), and a power series ring over A satisfies (f) if A does [8]. In this section, we show that these results are due to the fact that the relevant topological spaces are homeomorphic. In particular, a ring A inherits the property (f) from a power series ring, and any other property of rings which depends on the D-topology is also stable under polynomial and power series extension.

For the polynomial ring case, we generalize a little and work in the setting of weak content algebras. First we recall the definition of content module [5], [13].

Definition. Let M be an A-module. For $x \in M$, let $c(x)$ be the intersection of all ideals I of A for which $x \in IM$. We call M a *content* A-*module* if $x \in c(x)M$ for every $x \in M$. If $\phi: A \to B$ is an A-algebra which is a content A-module, then B is called a *weak content* A-*algebra* if for every prime ideal p of A, pB is either B or a prime ideal of B.

In [13] an A-algebra B is called a content A-algebra if B is a faithfully flat content A-module such that for every x, $y \in B$, the content formula

$$c(xy)\,c(y)^n = c(x)\,c(y)^{n+1}$$

holds, for some integer $n \geq 0$. The weak content algebra condition is equivalent to the condition that $c(x)\,c(y)$ be contained in the radical of $c(xy)$ for every x, $y \in B$. So it follows that every content A-algebra is a weak content A-algebra, but not conversely. The following lemma usually allows us to consider only the weak content A-algebras B for which $c(B) = A$.

LEMMA 3.1. *Let* B *be an* A-*algebra which is a content* A-*module. Then* $c(B) = c(1_B)$ *and is generated by an idempotent. Further,*

$$\{p \in \mathrm{Spec}(A) : pB = B\} = V(c(B)),$$

and thus is open and closed.

Proof. If $x \in B$, then writing $x = \sum a_i b_i$ ($a_i \in c(x)$, $b_i \in B$), we get that $c(x) = (a_1, \cdots, a_n)$. It follows that $c(xy) \subset c(x)\,c(y)$ for every x, $y \in B$. In particular, $c(1_B) \subset c(1_B)^2$. It now follows that $c(1_B) = eA$, where e is an idempotent. But $b \in B \Rightarrow c(b) = c(b \cdot 1_B) \subset c(b) \cdot c(1)$. Thus $c(B) = c(1_B)$. The last statement follows from

$$pB = B \Leftrightarrow 1 \in pB \Leftrightarrow c(1) \subset p.$$

Let $\phi: A \to B$ be an A-algebra and denote the map $\mathrm{Spec}(\phi): \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by $u: Y \to X$. If there is a continuous map $v: X \to Y$ such that for every $p \in X$ and $q \in Y$, $u \circ v(p)$ is D-related to p and $v \circ u(q)$ is D-related to q, then it follows that Y/D is homeomorphic to X/D. This is essentially what happens if B is either a weak content A-algebra, or a power series ring over A.

THEOREM 3.2. *If* $\phi: A \to B$ *is a weak content* A-*algebra, then* Y/D *is homeomorphic to an open-closed subset of* X/D.

Proof. Let $c(B) = eA$, where e is an idempotent of A. Since $eA \to eB = B$ is a weak content eA-algebra and Spec(eA) is an open-closed subset of X, we may assume $c(B) = A$, and show Y/D is homeomorphic to X/D. Then pB is a prime ideal of B for every prime ideal p of A. Let $v: X \to Y$ be defined by $v(p) = pB$. We will show that v is an inverse to u modulo the D-relation. To show v is continuous under the Zariski topologies, let $D(b) = \{y \in Y: b \notin y\}$ be a basic open set of Y and suppose $pB \in D(b)$. Then $b \notin pB \Rightarrow c(b) \not\subset p$. Let $t \in c(b)\setminus p$. Then $p \in D(t)$ and $v(D(t)) \subset D(b)$. If $t \notin q$, then $c(b) \not\subset q \Rightarrow b \notin qB$. Thus v is continuous and so induces a continuous map $v_D: X/D \to Y/D$. If $p \in X$, then

$$u \circ v(p) = pB \cap A \supset p,$$

so $u \circ v(p)$ and p are D-related. If $q \in Y$, then $v \circ u(q) = (q \cap A)B \subset q$, so q and $v \circ u(q)$ are D-related.

THEOREM 3.3. *If* B *is a power series ring over* A *in any set of indeterminates, then* $Y/D \cong X/D$.

Proof. Here we define $v: X \to Y$ by

$$v(p) = p[[X]] = \{b \in B: \text{the coefficients of } b \text{ are in } p\}.$$

If $v(p) = p[[X]] \in D(b)$ $(b \in B)$, then $b \notin p[[X]] \Rightarrow t \notin p$ for some coefficient t of b. Then $p \in D(t)$ and $v(D(t)) \subset D(b)$, for if $q \in D(t)$,

$$t \notin q \Rightarrow b \notin q[[X]] \Rightarrow v(q) \in D(b).$$

Further, for $p \in X$, $u \circ v(p) = p[[X]] \cap A = p$, and for $q \in Y$, $v \circ u(q) = (q \cap A)[[X]]$. Thus we must show that q and $(q \cap A)[[X]]$ are D-related, and for this it will suffice to show that they are not comaximal. If $b_1 + b_2 = 1$ $(b_1 \in q,\ b_2 \in (q \cap A)[[X]])$, let c_1 and c_2 be the constant terms of b_1 and b_2, respectively. Then

$$c_1 + c_2 = 1 \Rightarrow 1 - c_1 = c_2 \in q \Rightarrow b_1 + (1 - c_1) \in q.$$

But this is impossible, since $b_1 + (1 - c_1)$ has constant term 1 and hence is a unit of B.

PROPOSITION 3.4. *Let* H *be the functor from the category of commutative rings to the category of topological spaces defined by* $H(A) = \mathrm{Spec}(A)/D$. *If* $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ *are ring homomorphisms such that* $H(\psi\phi): H(C) \to H(A)$ *is a homeomorphism and* $H(\psi)$ *is surjective, then* $H(\phi)$ *is a homeomorphism.*

Proof. This is a consequence of the following lemma.

LEMMA. *If* $Z \xrightarrow{v} Y \xrightarrow{u} X$ *are morphisms in some category with* v *an epimorphism and* $u \circ v$ *an isomorphism, then* u *is an isomorphism.*

Proof. Let $w: X \to Z$ be the inverse of $u \circ v$. Then

$$1_X = (u \circ v) \circ w = u \circ (v \circ w);$$

$1_Z = w \circ (u \circ v)$ implies $v = v \circ w \circ u \circ v$, and then v an epimorphism implies $1_Y = v \circ w \circ u$. Thus $v \circ w$ is the inverse of u.

It follows from Theorems 3.2 and 3.3 and Proposition 3.4 that, for instance, if $A \subset B \subset C$ are subrings and C is either a weak content A-algebra or a power series ring over A, then B satisfies (f) if A does.

4. FINITE PRESENTATION OF FLAT A-ALGEBRAS

It is well known that a finitely presented flat nette A-algebra B is étale [14, p. 55, Corollary 1]. In [4] it is shown that a flat nette A-algebra B is formally étale, and an example is given showing that B is not necessarily étale because it may not be finitely presented. The following result shows that the condition (F) plays a role in this.

THEOREM 4.1. *Etale* A-*algebras satisfy* (F) *if and only if flat nette* A-*algebras are étale.*

Proof. (⇒) Let B be a flat nette A-algebra. To show B is étale, it suffices to show that for every prime q of B there exists $a \in B - q$ such that B_a is étale over A [14, p. 17, Proposition 6]. Then B_a is nette over A [14, p. 13, Proposition 1], so we have an exact sequence $0 \to I \to G \to B_a \to 0$, where G is an étale A-algebra [14, p. 51, Theorem 1, and p. 13, Proposition 1]. But since B_a is A-flat and G is étale, B_a is G-flat [7, p. 9, Proposition 2.7]. Thus I is a finitely generated ideal of G. Since G is finitely presented, it follows that B_a is finitely presented over A and thus étale.

(⇐) Let B be an étale A-algebra and I a pure ideal of B. Then B/I is flat and nette over A and thus finitely presented over A. Thus I is finitely generated [7, p. 49, F.4, F.5].

As the above result indicates, the condition (F) is intimately connected to the finite presentation of A-algebras. In [6, Lemma 2.1] it was shown that if every flat A-algebra generated by a single element over A is finitely presented, then every such A-algebra satisfies (F). The same argument shows that if every finite type flat A-algebra is finitely presented, then every finite type flat A-algebra satisfies (F). We do not know whether the converse of this statement holds, but the above theorem could be considered a partial result in this direction. The class of rings A for which every finite type flat A-algebra is finitely presented is known to contain all rings with finitely many minimal primes [15, p. 25, Theorem 3.4.6] and to be closed under taking subrings [L. Gruson, unpublished]. This is also clearly the case for the class of rings A for which every finite type flat A-algebra satisfies (F).

The condition (F) is more easily inherited by projective A-algebras than by flat ones. For example:

PROPOSITION 4.2. (1) *If* A *satisfies* (F), *then any projective* A-*algebra which is generated by one element satisfies* (F).

(2) (S. H. Cox) *If* A *is quasilocal, then any finite type projective* A-*algebra satisfies* (F).

Proof. (1) If $0 \to I \to A[X] \to B \to 0$ is exact with B projective, then the content c(I), being a pure ideal of A [12, Corollary 1.3], is generated by an idempotent e of A. By considering the ring eA, we may reduce to the case $e = 1$. But then B satisfies (F) by [12, Theorem 4.9].

(2) Let B be a finite type projective A-algebra and let m be the maximal ideal of A. If $b \in B$ is such that $b + mB$ is a regular element of B/mB, then b is regular in B [15, p. 16, Lemma 3.16]. Thus, if we let S be the set of $s \in B$ for which $s + mB$ is regular in B/mB, then S consists of regular elements and is a finite union of primes, since B/mB is noetherian. So $S^{-1}B$ is quasisemilocal and B satisfies (F).

Remark. The argument in Proposition 4.2-(2) can be extended from the quasilocal case to the case that A/J is artinian, where J is the Jacobson radical of A.

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