NAKANO'S THEOREM REVISITED

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This paper is a complement of the authors' paper [2]. In that paper we have provided a new proof of the following theorem (due to H. Nakano): *If* (L, τ) *is a Dedekind complete Riesz space with the Fatou property, then the order intervals of* L *are* τ*-complete.* (For a discussion and the history of this result, the reader is referred to [3], [5], [6], [7], and to the references of [2].) In the course of our proof of Nakano's theorem we made use of a nonelementary result. In this note we show, however, that it is possible to modify the proof of Nakano's theorem as it was presented in [2] so that it becomes elementary. On the other hand, an alternate proof of the same result will also be given.

For notation and basic terminology concerning Riesz spaces we refer the reader to [4]. A *locally solid Riesz space* (L, τ) is a Riesz space L equipped with a locally solid topology τ; that is, equipped with a linear topology τ which has a basis for zero consisting of solid sets. (A subset V of L is said to be a *solid set* if $|u| \leq |v|$ and $v \in V$ implies $u \in V$.) A net $\{u_\alpha\}$ of a Riesz space L *order converges to* u in L, denoted by $u_\alpha \xrightarrow{(o)} u$, if there exists a net $\{v_\alpha\}$ of L (with the same indexing set) such that $|u_\alpha - u| \leq v_\alpha \downarrow \theta$ holds in L. A subset V of a Riesz space is said to be *order closed* if $\{u_\alpha\} \subset V$ and $u_\alpha \xrightarrow{(o)} u$ implies $u \in V$, and V is said to have the *Fatou property* if V is solid and order closed. Note that a solid subset V of a Riesz space L is order closed if and only if $\theta \leq u_\alpha \uparrow u$ in L and $\{u_\alpha\} \subset V$ implies $u \in V$ (see [2, p. 25]).

A locally solid Riesz space (L, τ) has the Fatou property if τ has a basis for zero consisting of sets with the Fatou property. The *topological completion* $(\hat{L}, \hat{\tau})$ of a Hausdorff locally solid Riesz space (L, τ) equipped with the cone formed by the closure of L^+ in $\hat{L}$ is a locally solid Riesz space containing L as a Riesz subspace (see [1, Theorem 2.1, p. 109]).

A Riesz subspace L of a Riesz space K is said to be *order dense* in K if $\sup\{v \in L: \theta \leq v \leq u\} = u$ holds in K for all $u \in K^+$. If K is Archimedean this is equivalent to the property that for each $\theta < u \in K$ ($\theta < u$ means, of course, $\theta \leq u$ and $u \neq \theta$), there exists $v \in L$ with $\theta < v \leq u$. In particular, it follows that if L is order dense in K, the embedding of L into K preserves arbitrary suprema and infima.

We continue with a simple but very useful result.

LEMMA. *Assume that* L *is an order dense Riesz subspace of a Riesz space* K. *If* L *is a Dedekind complete Riesz space, then* L *is an ideal of* K.

Proof. Assume $\theta \leq u \leq v$ with $v \in L$ and $u \in K$. Pick a net $\{u_\alpha\} \subset L^+$ with $\theta \leq u_\alpha \uparrow u$ in K and notice that since L is Dedekind complete, $u_\alpha \uparrow w$ holds in L for some $w \in L^+$. But since L is order dense in K, $u_\alpha \uparrow w$ holds also in K. Hence $u = w \in L$; L is an ideal of K.

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The proof of Nakano's theorem given in [2] would be considered elementary if we could avoid a result stating that, in a Hausdorff locally solid Riesz space with the Fatou property, the Fatou subsets of the space are topologically closed. Although this result seems natural its proof is not elementary.

We present next two proofs of Nakano's theorem (for the Hausdorff case). The first one is a modification of the proof of [2] and the second an alternate one.

THEOREM (Nakano). *If* (L, τ) *is a Hausdorff Dedekind complete locally solid Riesz space with the Fatou property, then the order intervals of* L *are* τ*-complete.*

Proof. We have to show that L is an ideal of $\hat{L}$. Since $L^d = \{\theta\}$ in $\hat{L}$, according to the lemma it is enough to show that L is order dense in A(L), the ideal generated by L in $\hat{L}$. To this end, let $\theta < \hat{u} \leq u \in L$. Pick a sequence of Fatou neighborhoods of zero $\{V_n\}$ of (L, τ) with $V_{n+1} + V_{n+1} \subset V_n$ for $n = 1, 2, \cdots$ and with $\hat{u} \notin \overline{V}_1$, the $\hat{\tau}$-closure of V_1 in $\hat{L}$. Then select a net $\{v_\alpha\}$ of L, $\theta \leq v_\alpha \leq u$ for all $\alpha \in \{\alpha\}$, with $v_\alpha \xrightarrow{\hat{\tau}} \hat{u}$. In particular, note that $\{v_\alpha\}$ is a τ-Cauchy net of (L, τ). Next pick a sequence of indices $\{\alpha_n\} \subset \{\alpha\}$ with $\alpha_n \leq \alpha_{n+1}$ for $n = 1, 2, \cdots$, such that $v_\alpha - v_\beta \in V_{n+2}$ and $\hat{u} - v_\alpha \in \overline{V}_{n+1}$ for $\alpha, \beta \geq \alpha_n$. Put $u_n = v_{\alpha_n}$, $n = 1, 2, \cdots$, and note that $u_{n+p} - u_n \in V_{n+2}$ for $n, p = 1, 2, \cdots$. Note also that

$$\theta \leq \sup\{u_m: n \leq m \leq n+p\} - u_n = \sup\{u_m - u_n: n \leq m \leq n+p\}$$

$$\leq \sup\{|u_m - u_n|: n \leq m \leq n+p\}$$

$$\leq \sum_{m=n}^{n+p-1} |u_{m+1} - u_m| \in V_{n+2} + \cdots + V_{n+p+1} \subset V_{n+1}$$

for $n, p = 1, 2, \cdots$. Now put $w_n = \sup\{u_m: m \geq n\}$, $n = 1, 2, \cdots$, in L and note that since V_{n+1} is order closed in L, the above relation implies

$$\theta \leq w_n - u_n \in V_{n+1}$$

for $n = 1, 2, \cdots$. Observe next that $w_n \downarrow w \geq \theta$ in L and so

$$|w_{n+p} - u_n| \xrightarrow[(p \to \infty)]{(o)} |w - u_n| \quad \text{in } L.$$

But since $|w_{n+p} - u_n| \leq |w_{n+p} - u_{n+p}| + |u_{n+p} - u_n| \in V_{n+1}$ for $n, p = 1, 2, \cdots$, $|w - u_n| \in V_{n+1}$ for $n = 1, 2, \cdots$. Hence $|\hat{u} - w| \in \overline{V}_n$ for all n. (Note that $w = \limsup u_n$; the same holds for $\liminf u_n$.)

Now let

$$S = \left\{ v \in L^+ : v - w \in A = \bigcap_{n=1}^{\infty} V_n \right\}.$$

Obviously S is nonempty. Put $s = \inf S$ in L and notice that since A is a band of L, $s - w \in A$ and so $s > \theta$. Now choose a Fatou neighborhood of zero W_1 of (L, τ) such that $W_1 \subset V_1$. Pick a sequence of Fatou neighborhoods of zero $\{W_n\}$ of L with $W_{n+1} + W_{n+1} \subset W_n$ and $W_n \subset V_n$ for $n = 1, 2, \cdots$. Next choose a sequence of indices $\{\beta_n\} \subset \{\alpha\}$ with $\alpha_n \leq \beta_n \leq \beta_{n+1}$, for $n = 1, 2, \cdots$, such that $v_\alpha - v_\beta \in W_{n+2}$ and $\hat{u} - v_\beta \in \overline{W}_{n+1}$ for $\alpha, \beta \geq \beta_n$. (A simple verification shows that such a sequence exists.) Put $x_n = v_{\beta_n}$, $n = 1, 2, \cdots$, and use the above argument to get that $x = \limsup x_n$ in L satisfies $|\hat{u} - x| \in \overline{W}_n \subset \overline{V}_n$ for $n = 1, 2, \cdots$. Since

$$|x - w| \leq |x - x_n| + |x_n - u_n| + |u_n - w|,$$

$x - w \in V_n$ for all n; that is, $x - w \in A$. Thus $s \leq x$ and so

$$\theta \leq (s - \hat{u})^+ \leq (x - \hat{u})^+ \in \overline{W}_1 .$$

But then by [2, Lemma 3.1, p. 28], $(s - \hat{u})^+ \in \overline{W}_1$ for all the Fatou neighborhoods W_1 of zero of L with $W_1 \subset V_1$. Hence $(s - \hat{u})^+ = \theta$; that is, $\theta < s \leq \hat{u}$, and the proof is finished.

Alternate proof. Assume again $\theta < \hat{u} \leq u \in L$. Pick a Fatou neighborhood V_1 of zero of L with $\hat{u} \notin \overline{V}_1$ and choose a sequence of Fatou neighborhoods $\{V_n\}$ of zero of (L, τ) with $V_{n+1} + V_{n+1} \subset V_n$ for $n = 1, 2, \cdots$. Note that $A = \bigcap_{n=1}^\infty V_n$ is a band of L and, since L is Dedekind complete, a projection band of L. Thus $L = A \oplus A^d$ and by [1, Lemma 7.1, p. 122], $\hat{L} = \overline{A} \oplus (\overline{A})^d = \overline{A} \oplus \overline{A^d}$. Write $\hat{u} = \hat{u}_1 + \hat{u}_2$ with $\theta \leq \hat{u}_1 \in \overline{A}$ and $\theta \leq \hat{u}_2 \in (\overline{A})^d = \overline{A^d}$ and observe that $\theta < \hat{u}_2 \leq \hat{u} \leq u$. Now pick a net $\{v_\alpha\} \subset A^d$, $\theta \leq v_\alpha \leq u$ for all α, with $v_\alpha \xrightarrow{\hat{\tau}} \hat{u}_2$. Choose a sequence of indices $\{\alpha_n\} \subset \{\alpha\}$ with $\alpha_n \leq \alpha_{n+1}$ for $n = 1, 2, \cdots$, and $v_\alpha - v_\beta \in V_{n+2}$, $\hat{u}_2 - v_\alpha \in \overline{V}_{n+1}$ for $\alpha, \beta \geq \alpha_n$. Note that $w = \limsup v_{\alpha_n}$ in L satisfies $w \in A^d$ and $\hat{u}_2 - w \in \overline{V}_n$ for all n. Hence $\hat{u}_2 - w \in \overline{A^d} = (\overline{A})^d$. We will now show that $\hat{u}_2 - w \in \overline{A}$. To this end, let W_1 be a Fatou neighborhood of zero of L with $W_1 \subset V_1$. Choose a sequence of Fatou neighborhoods of zero $\{W_n\}$ of L with $W_{n+1} + W_{n+1} \subset W_n$ and $W_n \subset V_n$ for $n = 1, 2, \cdots$. Now proceed as above to select an element $x \in L$ with $\hat{u}_2 - x \in \overline{W}_n$ for all n and with $w - x \in A$. In particular, it follows that

$$(\hat{u}_2 - w) - (x - w) = \hat{u}_2 - x \in \overline{W}_1 ,$$

which implies that $\hat{u}_2 - w \in \overline{A}$. Hence $\hat{u}_2 - w \in \overline{A} \cap (\overline{A})^d = \{\theta\}$, and so $\hat{u}_2 = w \in L$. Thus $\theta < w \leq \hat{u}$ and the second proof is finished.

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