LACUNARY POWER SERIES ON THE UNIT CIRCLE

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By the statement that a formal power series

$$(1) \qquad S(\theta) = \sum_{n=1}^{\infty} c_n e^{ik_n\theta}$$

is q-*lacunary* we shall mean that its exponents k_n satisfy a condition of the form $k_{n+1}/k_n > q > 1$ $(n = 1, 2, \cdots)$. In a research announcement [2], R.E.A.C. Paley stated that *if the series* (1) *is* q-*lacunary, and if in addition* $|c_n| \to 0$ *and* $\sum |c_n| = \infty$, *then for each finite complex number* w *the series converges to* w *at every point of a set that is dense in* $[0, 2\pi]$.

A complete proof of Paley's theorem was later given by M. Weiss [3]. Subsequently, J.-P. Kahane, M. Weiss, and G. Weiss [1, pp. 1-16] showed that the plane-covering property of $S(\theta)$ is only one aspect of a much stronger property of the sequence $\{S_n\}$ of partial sums of (1). They proved that *if the series* (1) *is* q-*lacunary, and if in addition* $c_n \to 0$ *and* $\sum |c_n| = \infty$, *then corresponding to every closed connected subset* C *of the extended complex plane there exists an everywhere dense set* E *in* $[0, 2\pi]$ *such that for each* θ *in* E *the set* C *is the set of limit points of* $\{S_n(\theta)\}$.

This theorem fails if we omit the hypothesis that $c_n \to 0$. Indeed, let

$$E(\infty, S) = \{\theta \in [0, 2\pi] : \lim_{n\to\infty} |S_n(\theta)| = \infty\}.$$

If for each index n we take $c_n = n!$, then (even without the hypothesis of lacunarity) the series (1) obviously has the property that $E(\infty, S) = [0, 2\pi]$. It is not known in general whether the set $E(\infty, S)$ remains dense in $[0, 2\pi]$ if $c_n \not\to 0$. Simple arguments show that it is a dense set if we assume in addition that $q > 3$. In this note we prove the following result.

THEOREM. *To each* $q > 1$ *there corresponds a positive constant* A_q *such that for each* q-*lacunary series* (1) *satisfying the two conditions*

$$\limsup_{n\to\infty} |c_n| > 0$$

and

$$(2) \qquad \liminf_{N\to\infty} \left(\sum_{n=1}^{N} |c_n| \Big/ \max_{1\le n\le N} |c_n| \right) > A_q,$$

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the set $E(\infty, S)$ *is everywhere dense in* $[0, 2\pi]$.

Our proof of the theorem depends on the following lemma, which guarantees the existence of points θ in $[0, 2\pi]$ where $S_N(\theta)$ has the same order of magnitude as $\sum_1^N |c_n|$. Except for differences in notation, the lemma is Corollary (2.1) on page 6 of [1].

LEMMA. *To every* $q > 1$ *there correspond positive constants* B_q *and* C_q $(C_q > 1)$ *with the following property. If* (1) *is a* q-*lacunary power series and* w *is a complex number of modulus* $C_q \sum_{n=1}^N |c_n|$, *then each interval of length* B_q/n_1 *on the real line contains a point* ξ *such that*

$$|S_N(\xi) - w| \leq \sqrt{C_q^2 - 1} \sum_{n=1}^{N} |c_n| .$$

Proof of the Theorem. Let B_q and C_q be the constants of the lemma. We show that if x is a point in $[0, 2\pi]$, then for any $n_0 \geq 1$, there is a real number θ_0 such that

$$|x - \theta_0| \leq \frac{B_q}{2k_{n_0}(1 - 1/q)},$$

and

$$\lim_{N\to\infty} \left| \sum_{n=n_0}^{N} c_n e^{ik_n\theta_0} \right| = \infty .$$

Without loss of generality we may assume that $n_0 = 1$. In two steps we verify the theorem:

Step I. Let α and β be positive real numbers satisfying

$$\frac{B_q}{2C_q}\alpha + \frac{\sqrt{C_q^2 - 1}}{C_q} = 1 - \beta$$

and

$$C_q\beta < 1.$$

Express the series $S(\theta)$ as

$$S(\theta) = B_1(\theta) + B_2(\theta) + \cdots + B_i(\theta) + \cdots,$$

where

$$B_1(\theta) = c_1 e^{ik_1\theta} = c_1 e^{ik_{m_1}\theta}, \quad B_2(\theta) = \sum_{n=2}^{m_2} c_n e^{ik_n\theta}, \cdots,$$

$$B_i(\theta) = \sum_{n=m_{i-1}+1}^{m_i} c_n e^{ik_n\theta}, \cdots.$$

Choose the integers $m_1, m_2, \cdots$ by the conditions $m_1 = 1$, and m_i $(i \geq 2)$ is the smallest integer such that $m_i > m_{i-1}$ and

(3) $$k_1|c_1| + \cdots + k_{m_{i-1}}|c_{m_{i-1}}| \leq \alpha k_{m_{i-1}} \sum_{n=m_{i-1}+1}^{m_i} |c_n| .$$

Let

$$\sigma_1 = |c_1|, \quad \sigma_i = \sum_{n=m_{i-1}+1}^{m_i} |c_n| \quad (i = 2, 3, \cdots)$$

and

$$M_i = \max_{1 \leq n \leq m_i} |c_n| \quad (i = 1, 2, \cdots) .$$

For $i \geq 2$, the choice of m_i is such that

$$\sigma_i - |c_{m_i}| \leq (k_1|c_1| + \cdots + k_{m_{i-1}}|c_{m_{i-1}}|)/(\alpha k_{m_{i-1}}) .$$

This implies

$$\begin{aligned} \sigma_i &\leq |c_{m_i}| + M_i(k_1/k_{m_{i-1}} + \cdots + 1)/\alpha \\ &\leq M_i + M_i(1 + 1/q + 1/q^2 + \cdots)/\alpha \\ &= \left(1 + \frac{1}{\alpha(1 - 1/q)}\right) M_i . \end{aligned}$$

Thus we have

(4) $$\sigma_i \leq \left(1 + \frac{1}{\alpha(1 - 1/q)}\right) M_i \quad (i = 1, 2, \cdots) .$$

In particular, (4) shows that the σ_i are bounded if the c_n are bounded.

Now we define inductively a sequence $\{\theta_1 = x, \theta_2, \theta_3, \cdots\}$ such that

$$|\theta_{i+1} - \theta_i| \leq \frac{B_q}{2k_{m_i}}$$

and

$$|S_{m_i}(\theta_i)| \geq C_q \beta(\sigma_1 + \cdots + \sigma_i) .$$

Let $\theta_1 = x$. Clearly $|S_{m_1}(\theta_1)| = |c_1| \geq C_q\beta|c_1|$, since $C_q\beta < 1$. Having chosen θ_i, let $\phi_i = \arg(S_{m_i}(\theta_i))$ and let $w_i = C_q \sigma_{i+1} e^{i\phi_i}$. By the lemma, there exists a point θ_{i+1} in the interval $I = \left(\theta_i - \frac{B_q}{2k_{m_i}}, \theta_i + \frac{B_q}{2k_{m_i}}\right)$ such that

$$|B_{i+1}(\theta_{i+1}) - w_i| \leq \sqrt{C_q^2 - 1}\, \sigma_{i+1} .$$

Express $|S_{m_{i+1}}(\theta_{i+1})|$ as

$$|S_{m_{i+1}}(\theta_{i+1})| = |(S_{m_i}(\theta_i) + w_i) + (S_{m_i}(\theta_{i+1}) - S_{m_i}(\theta_i)) + (B_{i+1}(\theta_{i+1}) - w_i)| .$$

Incorporating the inductive hypothesis, we have

$$|S_{m_i}(\theta_i) + w_i| = |S_{m_i}(\theta_i)| + |w_i| \geq C_q \beta(\sigma_1 + \cdots + \sigma_i) + C_q \sigma_{i+1} .$$

Observe that for any real numbers a and b,

$$|e^{ia} - e^{ib}| = |e^{i(a-b)/2} - e^{-i(a-b)/2}| = \left|2i \sin\left(\frac{a-b}{2}\right)\right| \leq |a - b| . \tag{5}$$

Using (5), together with (3) and the definition of θ_{i+1}, we have

$$|S_{m_i}(\theta_{i+1}) - S_{m_i}(\theta_i)| \leq |\theta_{i+1} - \theta_i| \sum_{n=1}^{m_i} k_n |c_n| \leq \frac{\alpha B_q}{2} \sigma_{i+1} .$$

Thus

$$\begin{aligned} |S_{m_{i+1}}(\theta_{i+1})| &\geq C_q \beta(\sigma_1 + \cdots + \sigma_i) + C_q \sigma_{i+1} - \left\{\frac{\alpha B_q}{2C_q} + \frac{\sqrt{C_q^2 - 1}}{C_q}\right\} C_q \sigma_{i+1} \\ &= C_q \beta(\sigma_1 + \cdots + \sigma_{i+1}), \end{aligned}$$

completing the induction.

The sequence $\{\theta_i\}$ is a Cauchy sequence. For if $i < j$, then

$$|\theta_i - \theta_j| \leq |\theta_i - \theta_{i+1}| + \cdots + |\theta_{j-1} - \theta_j| \leq \frac{B_q}{2} \sum_{r=i}^{j-1} \frac{1}{k_{m_r}} \leq \frac{B_q}{2k_{m_i}(1 - 1/q)},$$

which tends to zero as $i \to \infty$. Since the sequence $\{\theta_i\}$ is then convergent, we define

$$\theta_0 = \lim_{i \to \infty} \theta_i .$$

The same argument shows

$$|\theta_i - \theta_0| \leq \frac{B_q}{2k_{m_i}(1 - 1/q)} \quad (i = 1, 2, \cdots) . \tag{6}$$

In particular,

$$|x - \theta_0| \leq \frac{B_q}{2k_1(1 - 1/q)} .$$

Step II. We assert that

$$\lim_{n \to \infty} |S_n(\theta_0)| = \infty .$$

If r is such that $m_i < r \leq m_{i+1}$, then

$$|S_r(\theta_0)| = \left|S_{m_i}(\theta_i) + (S_{m_i}(\theta_0) - S_{m_i}(\theta_i)) + \sum_{n=m_i+1}^{r} c_n e^{ik_n \theta_0}\right| .$$

From Step I,

$$|S_{m_i}(\theta_i)| \geq C_q \beta(\sigma_1 + \cdots + \sigma_i) .$$

Applying (5), together with (3) and (6), we have

$$|S_{m_i}(\theta_0) - S_{m_i}(\theta_i)| \leq |\theta_i - \theta_0| \sum_{n=1}^{m_i} k_n |c_n| \leq \frac{\alpha B_q}{2(1 - 1/q)} \sigma_{i+1} .$$

Thus

$$(7) \qquad \begin{aligned} |S_r(\theta_0)| &\geq C_q \beta(\sigma_1 + \cdots + \sigma_i) - \left(1 + \frac{\alpha B_q}{2(1 - 1/q)}\right) \sigma_{i+1} \\ &= C_q \beta(\sigma_1 + \cdots + \sigma_{i+1}) - \left(1 + \frac{\alpha B_q}{2(1 - 1/q)} + C_q \beta\right) \sigma_{i+1} . \end{aligned}$$

If the c_n are bounded, then $|S_r(\theta_0)|$ tends to infinity with $C_q \beta(\sigma_1 + \cdots + \sigma_{i+1})$, since $\sigma_{i+1} = O(1)$, by (4), and since $\limsup_{n \to \infty} |c_n| > 0$.

Otherwise, from (4) and (7), we have

$$(8) \qquad \begin{aligned} \frac{|S_r(\theta_0)|}{M_{i+1}} &\geq C_q \beta\left(\frac{\sigma_1 + \cdots + \sigma_{i+1}}{M_{i+1}}\right) - \left(1 + \frac{\alpha B_q}{2(1 - 1/q)} + C_q \beta\right) \frac{\sigma_{i+1}}{M_{i+1}} \\ &\geq C_q \beta\left(\frac{\sigma_1 + \cdots + \sigma_{i+1}}{M_{i+1}}\right) - \left(1 + \frac{\alpha B_q}{2(1 - 1/q)} + C_q \beta\right)\left(1 + \frac{1}{\alpha(1 - 1/q)}\right) . \end{aligned}$$

The last expression of (8) is bounded below by $\varepsilon > 0$ if

$$(9) \qquad \frac{\sigma_1 + \cdots + \sigma_{i+1}}{M_{i+1}} \geq \left\{\varepsilon + \left(1 + \frac{\alpha B_q}{2(1 - 1/q)} + C_q \beta\right)\left(1 + \frac{1}{\alpha(1 - 1/q)}\right)\right\} \Big/ C_q \beta .$$

We now define A_q by the formula

$$A_q = \left(1 + \frac{\alpha B_q}{2(1 - 1/q)} + C_q \beta\right)\left(1 + \frac{1}{\alpha(1 - 1/q)}\right) \Big/ C_q \beta .$$

Then condition (2) of the theorem implies that there is a positive ε such that (9) holds for all sufficiently large i. In this case, $|S_r(\theta_0)| \geq \varepsilon M_{i+1}$ and hence $\lim_{r \to \infty} |S_r(\theta_0)| = \infty$. This completes the proof of the theorem.

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