TRANSLATION-INVARIANT OPERATORS ON $L^p(G)$, $0 < p < 1$

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Let G be a compact abelian group, and for $0 < p \leq \infty$ let $L^p(G)$ denote the usual Lebesgue space with respect to normalized Haar measure on G. For $g \in G$ and functions f on G we define the translation operator T_g by $T_g f(h) = f(h - g)$ for $h \in G$. The collection $\{T_g : g \in G\}$ is a group of linear isometries on any $L^p(G)$, and we are interested in the bounded linear operators on $L^p(G)$ which commute with this group—the translation-invariant linear operators on $L^p(G)$. The problem of characterizing these operators is sometimes known as the multiplier problem and, for $p \geq 1$, has attracted much attention. Satisfactory characterizations are available only for the case $p = 1$ and the trivial case $p = 2$. Obtaining such a characterization for any other $p \geq 1$ appears to be a most difficult task, but for $p < 1$ the problem seems to have been neglected. The purpose of this note is to present such a characterization when $0 < p < 1$.

THEOREM. *Let* G *be a compact abelian group and fix* p *with* $0 < p < 1$. *The bounded linear operators on* $L^p(G)$ *which commute with each* T_g $(g \in G)$ *are precisely those operators of the form*

$$\sum_{i=1}^{\infty} a_i T_{g_i}, \quad \textit{where } g_i \in G \textit{ and } \sum_{i=1}^{\infty} |a_i|^p < \infty. \tag{1}$$

Proof. It is obvious that (1) defines a bounded and translation-invariant operator on $L^p(G)$. To show that each such operator is of the form (1), we require two lemmas.

LEMMA 1. *Let* K *be a compact Hausdorff space and let* λ *be a complex-valued regular Borel measure on* K. *If for some* p $(0 < p < 1)$ *and some finite positive number* M *we have*

$$\sum_{j=1}^{m} |\lambda(E_j)|^p \leq M \tag{2}$$

for each m *and each finite Borel partition* $\{E_j\}_{j=1}^m$ *of* K, *then* λ *is of the form* $\sum_{i=1}^{\infty} a_i \delta_{x_i}$, *where* δ_{x_i} *is the unit mass at some point* $x_i \in K$ *and* $\sum_{i=1}^{\infty} |a_i|^p \leq M$.

Proof. Assume first that λ is positive and let $\lambda = \lambda_d + \lambda_c$ be the decomposition of λ into discrete and continuous parts. Then (2) holds with either λ_d or λ_c in place of λ. If $\lambda_c(K) > 0$, then, as a consequence of [1, 11.44], for any $m = 1, 2, \cdots$ we can find disjoint Borel subsets $E_1, \cdots, E_m$ of K such that $\lambda_c(E_j) = m^{-1}\lambda_c(K)$, $j = 1, \cdots, m$. For these E_j we have

$$\sum_{j=1}^{m} |\lambda_c(E_j)|^p = m(m^{-1}\lambda_c(K))^p = m^{1-p}(\lambda_c(K))^p,$$

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and this contradicts (2) for λ_c as soon as $m^{1-p}(\lambda_c(K))^p > M$. Thus $\lambda_c(K) = 0$ and so $\lambda = \sum_{i=1}^{\infty} a_i \delta_{x_i}$, where $x_i \in K$ and $\sum_{i=1}^{\infty} |a_i| < \infty$. Now (2) implies that $\sum_{i=1}^{\infty} |a_i|^p < \infty$ as desired.

If λ is not positive, then it suffices to show that (2) holds when λ is replaced by its total variation measure $|\lambda|$. So fix a finite Borel partition $\{E_j\}_{j=1}^m$ of K and we will show that $\sum_{j=1}^m [|\lambda|(E_j)]^p \leq M$. By definition of $|\lambda|$, for each $\varepsilon > 0$ and $j = 1, \cdots, m$ there exists a finite Borel partition $\{E_i^j\}_{i=1}^{m_j}$ of E_j such that

$$|\lambda|(E_j) \leq \sum_{i=1}^{m_j} |\lambda(E_i^j)| + \left(\frac{\varepsilon}{m}\right)^{1/p}.$$

Then

$$\sum_{j=1}^{m} [|\lambda|(E_j)]^p \leq \sum_{j=1}^{m} \left(\sum_{i=1}^{m_j} |\lambda(E_i^j)| + \left(\frac{\varepsilon}{m}\right)^{1/p} \right)^p \leq \sum_{j=1}^{m} \sum_{i=1}^{m_j} |\lambda(E_i^j)|^p + \varepsilon \leq M + \varepsilon$$

by (2). Since $\varepsilon > 0$ was arbitrary, this establishes (2) for $|\lambda|$ and so completes the proof of the lemma.

LEMMA 2. *Let* G *be a compact abelian group and let* U *be a neighborhood of* 0 *in* G. *There exists a Borel set* E *contained in* U *such that for some finite subset* $\{g_1, \cdots, g_n\}$ *of* G, *the sets* E, $g_1 + E, \cdots, g_n + E$ *form a partition of* G.

Proof. Let T be the circle group. Considering products of half-open intervals in T, it is easy to see that the lemma holds if $G = T^m \times F$ for some finite group F and some $m = 1, 2, \cdots$. That is, the lemma is true when the character group Γ of G is a finitely generated group. We show that this implies the general case.

Let U be a neighborhood of 0 in G. Fix $\varepsilon > 0$ and characters $\gamma_1, \cdots, \gamma_k$ of G such that

$$\{g \in G: |\gamma_i(g) - 1| < \varepsilon, \ i = 1, \cdots, k\} \subseteq U.$$

Let G' be the character group of the subgroup Γ' of Γ generated by $\gamma_1, \cdots, \gamma_k$. Then there exist a continuous homomorphism ϕ of G onto G' and characters $\gamma_1', \cdots, \gamma_k'$ of G' such that $\gamma_i = \gamma_i' \circ \phi$ for $i = 1, \cdots, k$. Since the lemma holds for G', there exist a Borel subset E of G' and elements $g_1', \cdots, g_n'$ of G' such that

$$E \subseteq \{g' \in G': |\gamma_i'(g') - 1| < \varepsilon, \ i = 1, \cdots, k\}$$

and such that the sets E, $g_1' + E, \cdots, g_n' + E$ partition G'. Then the Borel sets $\phi^{-1}(E), \phi^{-1}(g_1' + E), \cdots, \phi^{-1}(g_n' + E)$ partition G, and

$$\phi^{-1}(E) \subseteq \{g \in G: |\gamma_i(g) - 1| < \varepsilon, \ i = 1, \cdots, k\} \subseteq U.$$

Finally, if $g_i \in G$ is selected so that $\phi(g_i) = g_i'$, $i = 1, \cdots, n$, then

$$\phi^{-1}(g_i' + E) = g_i + \phi^{-1}(E).$$

Thus the set $\phi^{-1}(E)$ and the elements $g_1, \cdots, g_n$ of G satisfy the conclusion of the lemma.

We return to the proof of the theorem. Let T be a bounded translation-invariant operator on $L^p(G)$. We will show that there exists a measure λ on G of the form

$$\lambda = \sum_{i=1}^{\infty} a_i \delta_{g_i}, \quad \text{where } g_i \in G \text{ and } \sum_{i=1}^{\infty} |a_i|^p \leq \sup_{0 \neq f \in L^p(G)} \int_G |Tf|^p \Big/ \int_G |f|^p$$

such that for each $f \in L^1(G)$, Tf is equal to $\lambda * f$, the convolution of λ and f. This will complete the proof of the theorem.

Since T commutes with translations, it is easy to check that if γ is in the character group Γ of G, then $T\gamma = c_\gamma \gamma$ for some constant c_γ. The boundedness of T implies that $\sup_{\gamma \in \Gamma} |c_\gamma| < \infty$, and so T is also a bounded linear operator on $L^2(G)$. It then follows from an extension of the Marcinkiewicz interpolation theorem due to Hunt [3] that T defines a bounded linear operator on $L^1(G)$. Hence it follows from Wendel's theorem [2, 35.5] that there is a complex-valued regular Borel measure on G such that $Tf = \lambda * f$ for $f \in L^1(G)$. We complete the proof by showing that λ satisfies the hypothesis (2) of Lemma 1 with

$$M = \sup_{0 \neq f \in L^p(G)} \int_G |Tf|^p \Big/ \int_G |f|^p . \tag{3}$$

Fix a Borel partition $\{E_j\}_{j=1}^m$ of G and an arbitrary $\varepsilon > 0$. Since λ is regular, there exist compact sets K_j with $K_j \subseteq E_j$ and pairwise disjoint open sets U_j with $K_j \subseteq U_j$, $j = 1, \cdots, m$, such that if F_j $(j = 1, \cdots, m)$ are any Borel sets satisfying $K_j \subseteq F_j \subseteq U_j$, then

$$\sum_{j=1}^{m} |\lambda(E_j) - \lambda(F_j)|^p < \varepsilon . \tag{4}$$

Let U be a neighborhood of 0 in G such that for each j we have $K_j + U - U \subseteq U_j$, and let E and $g_1, \cdots, g_n$ be as in Lemma 2 (where we take the present U). Since the sets E, $g_1 + E, \cdots, g_n + E$ partition G, the measure of E must be $(n + 1)^{-1}$. Thus

$$\int_G |\lambda(g + E)|^p \, dg \leq (n + 1)^{-1} M,$$

since $Tf = \lambda * f$ for $f \in L^1(G)$ and where M is defined by (3). Writing g_0 for $0 \in G$, it follows that

$$\int_G \sum_{i=0}^{n} |\lambda(g + g_i + E)|^p \, dg \leq M,$$

and so there exists some g (which we now fix) for which

$$\sum_{i=0}^{n} |\lambda(g + g_i + E)|^p \leq M .$$

For $i = 0, \cdots, n$, put $S_i = g + g_i + E$. Then the inequality above becomes

$$\sum_{i=0}^{n} |\lambda(S_i)|^p \leq M. \tag{5}$$

The collection $\{S_i\}_{i=0}^n$ is a partition of G, and since $E \subseteq U$ and $K_j + U - U \subseteq U_j$ for each j, it follows that if $S_i \cap K_j \neq \emptyset$, then $S_i \subseteq U_j$. For $j = 1, \cdots, m$ set

$$F_j = \bigcup_{S_i \cap K_j \neq \emptyset} S_i .$$

Then the F_j are Borel sets and $K_j \subseteq F_j \subseteq U_j$ for each j. Thus it follows from (4) and (5) that

$$\sum_{j=1}^{m} |\lambda(E_j)|^p \leq \sum_{j=1}^{m} |\lambda(E_j) - \lambda(F_j)|^p + \sum_{j=1}^{m} \Big| \sum_{S_i \cap K_j \neq \emptyset} \lambda(S_i)\Big|^p$$

$$\leq \varepsilon + \sum_{j=1}^{m} \sum_{S_i \cap K_j \neq \emptyset} |\lambda(S_i)|^p \leq \varepsilon + \sum_{i=0}^{n} |\lambda(S_i)|^p \leq \varepsilon + M.$$

Since $\varepsilon > 0$ and the partition $\{E_j\}_{j=1}^m$ were arbitrary, this establishes the hypothesis (2) of Lemma 1 for λ. The proof of the theorem is complete.

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