A NOTE ON DIRECT INTEGRALS OF SPECTRAL OPERATORS

Edward A. Azoff

1. INTRODUCTION

It is easy to see that a decomposable operator is normal if and only if almost all of its direct integrands are normal. The following theorem, stated by T. R. Chow in [1], provides a corresponding characterization of decomposable spectral operators.

THEOREM 1.1. *Let* A *be a bounded decomposable operator on the separable Hilbert space* H,

$$A = \int_{\Sigma}^{\oplus} A(s)\, d\mu(s).$$

Then A *is spectral if and only if*

(i) A(s) *is spectral for* μ*-almost every* s, *with spectral measure* E_s,

(ii) *for each Borel set* $B \subseteq \mathbb{C}$, *the function* $S \mapsto E_s(B)$ *is measurable on* Σ,

(iii) $\sup\{\mu\text{ - ess sup } \|E_s(B)\|;\ B$ *a Borel set in* $\mathbb{C}\} < \infty$,

(iv) $\lim(\mu\text{ - ess sup } \|N(s)^n\|^{1/n}) = 0$, *where* N(s) *is the radical part of* A(s).

A gap in Chow's proof was filled by M. J. J. Lennon in [6]. Lennon also gave examples to show that the theorem would no longer be true if either of the conditions (iii) or (iv) were omitted. On the other hand, both Chow and Lennon conjectured that condition (ii) is redundant. In Section 2 of this note, we establish this conjecture, essentially by reducing the problem to the (known) case of normal operators. As a by-product of this technique, we show in Section 3 that every direct integral of spectral operators is in fact a direct sum of spectral operators.

We use the notation established in [6]. In particular, all operators discussed will be bounded operators acting on separable Hilbert spaces, and we follow Dixmier's formulation of direct-integral theory [2]. A complete discussion of spectral operators can be found in [3]. Beyond the basic definitions, the main fact needed below is the canonical decomposition of spectral operators; this material can be found in the first four sections of [3].

2. REDUNDANCY OF CONDITION (ii)

The main result to be proved in this section is the following theorem.

THEOREM 2.1. *Let* $s \mapsto A(s)$ *be a measurable field of spectral operators, and write* E_s *for the spectral measure of* A(s). *Then for each Borel set* B *the field* $s \mapsto E_s(B)$ *is measurable.*

Received August 27, 1975.
The research for this paper was partially supported under NSF grant NPS 75-07901.

Michigan Math. J. 23 (1976).

The reason this theorem is not an immediate application of von Neumann's measurable-choice principle is that there is no obvious "countable process" for constructing E_s from $A(s)$. Our first lemma, which is well known, shows that for fields of normal operators this is not a problem.

LEMMA 2.2. *Theorem* 2.1 *is true if all the* $\{A(s)\}$ *are normal.*

Proof. Since $\|A(s)\|$ depends measurably on s, we may assume that the $\{A(s)\}$ have norms uniformly bounded by some constant K. For each polynomial p in z and $\bar{z}$, the field $s \mapsto p(A(s), A^*(s))$ is measurable. If B is a compact subset of the plane, then there is a sequence $\{p_n\}$ of such polynomials that are uniformly bounded on $\{z \mid |z| \leq K\}$ and converge pointwise to the characteristic function of B on $\{z \mid |z| \leq K\}$. Thus $E_s(B)$ is the weak limit of the $p_n(A(s), A^*(s))$, and the field $s \mapsto E_s(B)$ is measurable for each compact B.

Now let $\mathscr{B}$ be the collection of subsets B of the plane such that the map $s \mapsto E_s(B)$ is measurable. Clearly $\mathscr{B}$ is closed under countable unions and complementation. Thus $\mathscr{B}$ contains the σ-algebra generated by the compact sets. ∎

It was shown by J. Wermer [7] that every spectral operator of scalar type is similar to a normal operator. The following lemma records several observations that are implicit in Wermer's proof.

LEMMA 2.3. *Let* S *be a scalar spectral operator with spectral measure* E. *Then there exists an invertible operator* T *in the von Neumann algebra generated by* S *such that* TST^{-1} *is normal. We always have the inequality* $\|E\| \leq \|T\| \, \|T^{-1}\|$, *and it is possible to make both* $\|T\|$ *and* $\|T^{-1}\|$ *bounded by* $2\|E\|$.

Proof. By Lemma 1 of [7], for each Borel partition $\pi = \{\sigma_i\}_{i=1}^n$ of the plane and each $x \in H$, we have the relations

$$\frac{1}{4\|E\|^2}\|x\|^2 \leq \sum_{i=1}^{n} \|E(\sigma_i)x\|^2 \leq 4\|E\|^2 \|x\|^2 .$$

Set $A_\pi = \sum E(\sigma_i)^* E(\sigma_i)$. Then A_π is self-adjoint, and it satisfies the condition $\frac{1}{4\|E\|^2} \leq A_\pi \leq 4\|E\|^2$. Since the unit ball of $\mathscr{L}(H)$ is weakly compact, the A_π have a weak limit point A. Clearly, A continues to satisfy the inequalities $\frac{1}{4\|E\|^2} \leq A \leq 4\|E\|^2$. Moreover, since $A_\pi E(\sigma) \geq 0$ for each partition π refining $\{\sigma, \sigma^c\}$, we see that $AE(\sigma) \geq 0$ for every Borel set σ.

Set $T = A^{1/2}$. Then $\frac{1}{2\|E\|} \leq T \leq 2\|E\|$, so that the last assertion of the lemma is verified. Also, each $E(\sigma)$ belongs to the second commutant of S. Thus the $\{A_\pi\}$ all belong to the von Neumann algebra generated by S, and hence so does T.

Let $F(\sigma) = TE(\sigma)T^{-1}$. Then $F(\cdot)$ is a spectral measure that is self-adjoint, since $(F(\sigma)x, x) = (AE(\sigma)T^{-1}x, T^{-1}x) \geq 0$ for each $x \in H$. That TST^{-1} is normal follows from the equation $TST^{-1} = \int \lambda \, dF$, and the relation $T^{-1}F(\cdot)T = E(\cdot)$ shows that $\|E\| \leq \|T\| \, \|T^{-1}\|$. ∎

Now let $s \mapsto A(s)$ be a measurable field of spectral operators, and consider the system of equations

$$(*) \qquad \begin{cases} A(s) = S + N, \\ S = XLY. \end{cases}$$

Here all operators except $A(s)$ are unknowns: N is quasi-nilpotent, S and N commute, X and Y are inverses of each other, and L is normal. In view of the previous lemma and the canonical decomposition of spectral operators, (*) always has at least one solution. In fact, S and N are uniquely determined by $A(s)$; but in general, X and Y are not. By the *norm* of a solution to (*), we mean the maximum of norms of the operators involved (including $A(s)$).

LEMMA 2.4. *For each integer* K, *the set* Σ_K *of* s *in* Σ *for which* (*) *has a solution of norm at most* K *is measurable. Moreover, there exist measurable fields* $S(\cdot)$, $N(\cdot)$, $X(\cdot)$, $L(\cdot)$, $Y(\cdot)$ *of solutions to* (*) *that are bounded on each* Σ_K.

Proof. Since the measurability of a field is not affected when the field is changed on a set of measure zero, we may assume that $A(s)$ depends Borel-measurably on s. Also, in view of [2, Proposition 3, page 145], we can restrict attention to the case where the underlying field of Hilbert spaces is the constant field corresponding to some space H_0.

Let G_K denote the set of 6-tuples (s, S, N, X, L, Y) such that (*) holds for $A(s)$ and constitutes a solution of norm at most K. Since multiplication and adjunction are Borel maps on bounded subsets of $\mathscr{L}(H_0)$, we see that G_K is a Borel set. By the principle of measurable choice [2, p. 332], $\pi_1(G_K)$ is measurable, and there exist measurable fields $S_K(\cdot)$, $N_K(\cdot)$, $X_K(\cdot)$, $L_K(\cdot)$, and $Y_K(\cdot)$ whose common graph lies in G_K. But because $\pi_1(G_K)$ coincides with Σ_K, the first statement of the lemma is established.

Set $S(s) = \begin{cases} S_1(s) & \text{if } s \in \Sigma_1, \\ S_K(s) & \text{if } s \in \Sigma_K \setminus \Sigma_{K-1}, \end{cases}$ and define $N(\cdot)$, $X(\cdot)$, $L(\cdot)$, and $Y(\cdot)$ similarly. This completes the proof. ■

Proof of Theorem 2.1. Use the preceding lemma to write $A(s) = S(s) + N(s)$ and $S(s) = X(s)L(s)Y(s)$, all fields being measurable. Let F_s be the spectral measure of $L(s)$. Since E_s is also the spectral measure of $S(s)$, and $E_s = X(s)F_sY(s)$, an appeal to Lemma 2.2 completes the proof. ■

In particular, condition (ii) of Theorem 1.1 is redundant. We close this section with a variant of Theorem 1.1 that will be useful in Section 3. The norm estimates in Lemma 2.3 could be used to convert the following proof into a proof of Theorem 1.1.

THEOREM 2.5. *Let* $A = \int^{\oplus} A(s)\, d\mu(s)$ *be a direct integral of spectral operators, and choose measurable fields of operators as in* Lemma 2.4. *Then* A *is spectral if and only if*

(i) $\operatorname{ess\,sup} \|X(s)\| < \infty$,

(ii) $\operatorname{ess\,sup} \|Y(s)\| < \infty$,

(iii) $\lim_n \operatorname{ess\,sup} \|N(s)^n\|^{1/n} = 0$.

Proof. Suppose A is spectral. By Lemma 2.3, we can write $A = S + N$ and $S = T^{-1} L T$, where L is normal. Since every operator commuting with A commutes with both S and N, and T belongs to the von Neumann algebra generated by A, all operators involved are decomposable. In particular (neglecting a set of measure zero), there exists a uniformly bounded family of solutions to (*). Thus (i) and (ii) follow from Lemma 2.4, and (iii) follows from the fact that $\|N^n\| = \text{ess sup} \|N(s)^n\|$.

Conversely, assume (i), (ii), and (iii). Note that for each s, the operators L(s) and A(s) are similar and hence have the same spectrum. Since the norm of a normal operator is equal to its spectral radius, it follows that $\|L(s)\| \leq \|A(s)\|$. Thus all the fields constructed in Lemma 2.4 are essentially bounded, and it makes sense to form their direct integrals. Now $L = \int^{\oplus} L(s)\, d\mu(s)$ is normal, and condition (iii) means that $N = \int^{\oplus} N(s)\, d\mu(s)$ is quasi-nilpotent. It follows that $A = XLY + N$ is spectral. ■

3. DIRECT INTEGRALS VERSUS DIRECT SUMS

F. Gilfeather [5] has shown that every direct integral of quasi-nilpotents is a direct sum of quasi-nilpotents. By applying Theorem 2.5, we can extend his result to spectral operators.

THEOREM 3.1. *Every direct integral of spectral operators is a direct sum of spectral operators.*

Proof. Let $A = \int_{\Sigma}^{\oplus} A(s)\, d\mu(s)$ be a direct integral of spectral operators, and fix a field $s \mapsto A(s)$ representing A. Let the fields $S(\cdot)$, $X(\cdot)$, $N(\cdot)$, and $Y(\cdot)$ be chosen as in Lemma 2.4. Since each N(s) is quasi-nilpotent, the sequence of real-valued functions $s \mapsto \|N(s)^n\|^{1/n}$ converges pointwise to zero. By Egoroff's Theorem, we can find a set $E \subseteq \Sigma$ of arbitrarily small measure such that convergence is uniform off E. By taking E a little larger if necessary, we make the fields $s \mapsto X(s)$ and $s \rightarrow Y(s)$ uniformly bounded off E. Thus, by Theorem 2.5, $\int_{E^c}^{\oplus} A(s)\, d\mu(s)$ is spectral. A standard measure-theoretic exhaustion argument completes the proof. ■

As a particular instance of Theorem 3.1, we mention the fact that every operator A belonging to a finite von Neumann algebra of type I is a direct sum of spectral operators. Indeed, A has a direct-integral decomposition $A = \int_{\Sigma}^{\oplus} A(s)\, d\mu(s)$, where all the $\{A(s)\}$ act on finite-dimensional spaces and hence are spectral. The reader is referred to [4] for a discussion of a larger class of operators to which Theorem 3.1 applies.

REFERENCES

1. T. R. Chow, *A spectral theory for direct integrals of operators.* Math. Ann. 188 (1970), 285-303.

2. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien.* 2nd edition, Gauthier-Villars, Paris, 1969.

3. N. Dunford and J. T. Schwartz, *Linear operators. Part III: Spectral operators.* Wiley-Interscience, New York, 1971.

4. F. Gilfeather, *Operator valued roots of abelian analytic functions.* Pacific J. Math. 55 (1974), 127-148.

5. ———, *Reductive operators with a spectral separation property* (preprint).

6. M. J. J. Lennon, *Direct integral decomposition of spectral operators.* Math. Ann. 207 (1974), 257-268.

7. J. Wermer, *Commuting spectral measures on Hilbert space.* Pacific J. Math. 4 (1954), 355-361.

The University of Iowa, Iowa City, Iowa 52242
Current address: University of Georgia, Athens, Georgia 30602