H^p-DERIVATIVES OF BLASCHKE PRODUCTS

C. N. Linden

1. INTRODUCTION

A Blaschke product B is a function defined by a formula

$$(1.1)\qquad B(z, \{a_n\}) = z^m \prod_{a_n \neq 0} \frac{\bar{a}_n}{|a_n|}\left(\frac{a_n - z}{1 - z\bar{a}_n}\right),$$

where $\sum_n (1 - |a_n|) < \infty$, $|a_n| < 1$ for all n, and m is the number of zeros in the sequence $\{a_n\}$. D. Protas [4] has shown that if, in addition,

$$(1.2)\qquad \sum_n (1 - |a_n|)^\alpha = S < \infty$$

for some α in (0, 1/2), then $B' \in H^p$, that is, the integrals

$$(1.3)\qquad \int_0^{2\pi} |B'(re^{i\theta}, \{a_n\})|^p \, d\theta \qquad (0 < r < 1)$$

are bounded when $0 < p \leq 1 - \alpha$.

The work of O. Frostman [2, Theorem IX] shows that (1.2) with $\alpha > 1/2$ does not necessarily imply the boundedness of the integrals (1.3) on $0 < r < 1$ for all positive p.

In this paper we extend the theorem of Protas to higher-order derivatives as follows, and give some relevant counterexamples.

THEOREM 1. *Let* k *be a natural number, and let* $\{a_n\}$ *be a Blaschke sequence such that* (1.2) *holds for some* α *in* $\left(0, \frac{1}{k+1}\right)$. *Then, if* $m = (1 - \alpha)/k$, *there is a constant* C(m, k) *such that*

$$(1.4)\qquad \int_0^{2\pi} \left|\frac{B^{(k)}(re^{i\theta}, \{a_n\})}{B(re^{i\theta}, \{a_n\})}\right|^m d\theta < C(m, k)S \qquad (1/2 < r < 1).$$

Hence $B^{(k)} \in H^p$ *for each* p *in* (0, m].

At each subsequent appearance, the symbol C denotes a positive constant depending either explicitly or implicitly on the parameters indicated. However the value of C may vary from one appearance to the next.

Received July 7, 1975.

Michigan Math. J. 23 (1976).

2. THE PROOF OF THEOREM 1

A proof of Theorem 1 in the case $k = 1$ is included in Protas's proof of the boundedness of the integrals (1.3) when $0 < r < 1$. In order to prove the generalisation, we proceed by induction, and suppose that the conclusion of Theorem 1 is valid for $k = 1, 2, \cdots, \nu$. Let

$$0 < \alpha < \frac{1}{\nu + 2}, \tag{2.1}$$

and put $m = \frac{1 - \alpha}{\nu + 1}$. We suppose that r is fixed in (1/2, 1).

For convenience, we shall write $B(z)$ in place of $B(z, \{a_n\})$. Then the logarithmic derivative of B is given by the formula

$$\frac{B'(z)}{B(z)} = - \sum_n \frac{1 - r_n^2}{(1 - z\bar{a}_n)(a_n - z)},$$

where $r_n = |a_n|$. Thus the principle of induction can be used to establish the formula

$$\frac{B^{(\nu+1)}(z)}{B(z)} = \sum_{k=2}^{\nu+1} \sum{}' C(n_1, n_2, \cdots, n_k, \nu) \prod_{j=1}^{k} \frac{B^{(n_j)}(z)}{B(z)} - \sum_n \sum_{j=0}^{\nu} \frac{(1 - r_n^2)\,\bar{a}_n^j\, \nu!}{(1 - z\bar{a}_n)^{j+1} (a_n - z)^{\nu - j + 1}}, \tag{2.2}$$

where, for each k, the sum $\sum'$ is taken over all k-tuples $(n_1, n_2, \cdots, n_k)$ for which $n_1 + n_2 + \cdots + n_k = \nu + 1$ and $n_j \geq 1$ for $1 \leq j \leq k$. The coefficients $C(n_1, n_2, \cdots, n_k, \nu)$ are integers, not necessarily nonzero.

Let $S_1(z)$ and $S_2(z)$ denote respectively the two sums that make up the right-hand side of (2.2). Then $S_1(z)$ is a linear combination of terms of the form

$$T(z) = \prod_{j=1}^{k} \frac{B^{(n_j)}(z)}{B(z)} \qquad (2 \leq k \leq \nu + 1),$$

where $n_1 + n_2 + \cdots + n_k = \nu + 1$ and $n_j \geq 1$ for $1 \leq j \leq k$. Hence $n_j \leq \nu$ for $j = 1, 2, \cdots, k$, and Hölder's inequality implies

$$\int_0^{2\pi} |T(re^{i\theta})|^m \, d\theta \leq \prod_{j=1}^{k} \left\{ \int_0^{2\pi} \left| \frac{B^{(n_j)}(re^{i\theta})}{B(re^{i\theta})} \right|^{mt_j} d\theta \right\}^{1/t_j},$$

where $t_j = (\nu + 1)/n_j$. But then

$$n_j m t_j = m(\nu + 1) = 1 - \alpha,$$

and since (1.4) is assumed to be valid for $k = 1, 2, \cdots, \nu$, and since $\sum_{j=1}^{k} \frac{1}{t_j} = 1$, we obtain the inequality

$$\int_0^{2\pi} |T(re^{i\theta})|^m \, d\theta \le \prod_{j=1}^{k} C(m, t_j, n_j) S^{1/t_j} = C(m, n_1, n_2, \cdots, n_k, \nu) S.$$

The inequality $0 < m < 1$ now leads to

$$\int_0^{2\pi} |S_1(re^{i\theta})|^m \, d\theta < C(m, \nu) S. \tag{2.3}$$

In considering the sum $S_2(z)$, we need to examine terms of the form

$$\tau_{n,j}(z) = (1 - z\bar{a}_n)^{-j-1} (a_n - z)^{j-\nu-1} \qquad (0 \le j \le \nu),$$

for which we have the estimate

$$\int_0^{2\pi} |\tau_{n,j}(re^{i\theta})|^m d\theta \le 2 \int_0^{\pi} |1 - rr_n e^{i\theta}|^{-m} |re^{i\theta} - r_n|^{-m(\nu+1)} d\theta .$$

Now, if $r_n \le 1/4$, then $|re^{i\theta} - r_n| \ge r - r_n > 1/4$, while if $r_n > 1/4$, then

$$|re^{i\theta} - r_n|^2 \ge 4rr_n \sin^2 \theta/2 \ge \frac{1}{2}\left(\frac{\theta}{\pi}\right)^2$$

for $0 \le \theta < \pi$. Hence

$$|re^{i\theta} - r_n| \ge \theta/4\pi \qquad (0 \le r_n < 1,\ 0 \le \theta < \pi).$$

Similarly, by considering separately the cases in which $r_n \le 1/2$ and $r_n > 1/2$, we obtain the relations

$$|1 - rr_n e^{i\theta}| = ((1 - rr_n)^2 + 4rr_n \sin^2 \theta/2)^{1/2} > \frac{1 - r_n + \theta}{2 + \pi}$$

for $0 \le r_n < 1$ and $0 \le \theta < \pi$, and hence

$$\int_0^{2\pi} |\tau_{n,j}(re^{i\theta})|^m d\theta \le C(m, \nu) \int_0^{\pi} (1 - r_n + \theta)^{-m} \theta^{-m(\nu+1)} d\theta$$

$$\le C(m, \nu)(1 - r_n)^{1-m(\nu+2)},$$

because $m(\nu + 1) = 1 - \alpha < 1$.

Since for each n the terms of S_2 are finite linear combinations of terms of the form $(1 - r_n^2)\tau_{n,j}$, and $0 < m < 1$, we see that

$$\int_0^{2\pi} |S_2(re^{i\theta})|^m d\theta < C(m, \nu) \sum_n (1 - r_n)^{1-m(\nu+1)} = C(m, \nu) S .$$

Together with (2.3), this shows that (1.4) is valid for $k = \nu + 1$. Thus Theorem 1 is proved by induction.

3. COUNTEREXAMPLES

The following examples indicate various ways in which Theorem 1 may be regarded as being best possible.

Let b *be a Blaschke product such that*

$$b(z) = \prod_{n=1}^{\infty} \frac{r_n^{q_n} - z^{q_n}}{1 - r_n^{q_n} z^{q_n}} = \prod_{n=1}^{\infty} b_n(z), \tag{3.1}$$

where $q_n = p_n 2^{p_n}$ *and* $p_n = 2^{2^n}$ *for* $n \geq 1$, *and let* $\{a_n\}$ *denote the zeros of* b. *Let* M *be a natural number.*

(i) *If* $0 < \beta < \frac{1}{M+1}$ *and* $r_n = 1 - (2^{p_n})^{-1/\beta}$, *then*

$$I_m(r) = \int_0^{2\pi} |b^{(M)}(re^{i\theta})|^m d\theta$$

is unbounded on (0, 1) *for* $Mm = (1 - \beta)$, *while* $\sum_{n=1}^{\infty} (1 - |a_n|)^\alpha$ *converges if and only if* $\alpha > \beta$.

(ii) *If* $0 < \beta < \frac{1}{M+1}$, $r_n = 1 - (p_n^2 2^{p_n})^{-1/\beta}$, *and* $Mm > 1 - \beta$, *then the integrals* $I_m(r)$ *are unbounded on* (0, 1), *while* $\sum_{n=1}^{\infty} (1 - |a_n|)^\beta$ *converges.*

(iii) *If* $\beta = \frac{1}{M+1}$ *and* $r_n = 1 - (2^{p_n})^{-1/\beta}$, *then, for each positive number* m, *the integrals* $I_m(r)$ *are unbounded on* (0, 1), *while* $\sum_{n=1}^{\infty} (1 - |a_n|)^\alpha$ *converges if and only if* $\alpha > \beta$.

We note that P. R. Ahern and D. N. Clark [1, p. 122] have constructed an example of a Blaschke product that has the properties illustrated in (ii) in the case M = 1 and, in addition has its zeros converging to 1.

It follows immediately from our definitions that in cases (i) and (iii)

$$\sum_{n=1}^{\infty} (1 - |a_n|)^\alpha = \sum_{n=1}^{\infty} p_n 2^{p_n(1-\alpha/\beta)},$$

and the series on the right converges if and only if $\alpha > \beta$. In case (ii), we have the series

$$\sum_{n=1}^{\infty} (1 - |a_n|)^\beta = \sum_{n=1}^{\infty} p_n^{-1},$$

which clearly converges.

We can complete each of the cases (i), (ii), (iii) by showing that, for each of a sequence of values r increasing to 1, the function b can be factored so that the contribution to the integral $I_m(r)$ arising from one of the factors is dominant. Because of the similarity of the proofs, we shall establish only (i) and forego the corresponding details for (ii) and (iii). We continue by proving the following lemma, which enables us to make suitable estimates of the factors of b.

LEMMA 1. *Let* b *denote the Blaschke product defined by* (3.1) *in the case* (i). *For each natural number* t, *let*

$$B_1 = \prod_{n=1}^{t-1} b_n, \qquad B_2 = \prod_{n=t+1}^{\infty} b_n.$$

Then, for $r = r_t$, *we have the inequalities*

$$\int_0^{2\pi} |b_t^{(j)}(re^{i\theta})|^{(1-\beta)/j}\, d\theta \geq C(j, \beta)\, p_t, \tag{3.2}$$

$$\int_0^{2\pi} |b_t^{(k)}(re^{i\theta})B_1^{(j-k)}(re^{i\theta})|^{(1-\beta)/j}\, d\theta \leq C(j, k, \beta) p_t^{k/j}\, p_{t-1}^{(j-k)/j} \quad (k = 0, 1, \cdots, j), \tag{3.3}$$

$$|B_2^{(j)}(re^{i\theta})| < C(j, \beta)\, q_{t+1}^{j!} \exp(-p_{t+1}\, 2^{p_t(p_{t-1}/\beta)}) \qquad (0 \leq \theta < 2\pi) \tag{3.4}$$

for $j = 1, 2, \cdots, M$, *and*

$$1 \geq |B_1(z)| > 1 - C(\beta)\, 2^{(p_{t-1} - p_t)/\beta}, \tag{3.5}$$

$$1 \geq |B_2(z)| > 1 - C(\beta)\, q_{t+1}(1 - r_{t+1}^{q_{t+1}}), \tag{3.6}$$

for $|z| = r$.

For the proof of this lemma, we need the following standard result; see, for example, [3, pp. 92-96].

LEMMA 2. *If* $\alpha \in (0, 1) \cup (1, \infty)$, *and* μ *is a natural number, then*

$$C(\alpha) < (1 - r)^{K(\alpha)} \int_0^{2\pi} |1 - re^{i\mu\theta}|^{-\alpha}\, d\theta < C(\alpha) \qquad (0 < r < 1),$$

where $K(\alpha) = \max(0, \alpha - 1)$ *and* $C(\alpha)$ *does not depend on* μ.

In the proof of Lemma 1 it will be notationally convenient to write q and γ for q_t and $1/\beta$, respectively. Then we can generalise the derivative formula

$$b_t'(z) = \frac{q}{z}(r^q - r^{-q})(P(z)^2 - P(z)),$$

where

$$P(z) = (1 - r^q z^q)^{-1}, \qquad P'(z) = \frac{q}{z}(P(z)^2 - P(z)),$$

to obtain, by induction, the formula

$$b_t^{(j)}(z) = z^{-j}(r^q - r^{-q})Q_j(P(z), q),$$

where $Q_j(P(z), q)$ is a polynomial of degree $j + 1$ in $P(z)$ and of degree j in q with leading coefficient $j!$. Lemma 2 shows that the inequalities $1 \leq k \leq j \leq M$ imply

$$J_k = \int_0^{2\pi} |P(re^{i\theta})|^{k(1-\beta)/j}\, d\theta < C(k, j, \beta).$$

Further, if $2 \leq k = j + 1 \leq M + 1$, then

$$\frac{k(1-\beta)}{j} > \frac{M(j+1)}{(M+1)j} \geq 1,$$

and Lemma 2 implies

$$C(j, \beta)(1 - r^q)^{\beta-(1-\beta)/j} < J_{j+1} < C(j, \beta)(1 - r^q)^{\beta-(1-\beta)/j}.$$

Thus the inequality (3.2) follows, since

$$\int_0^{2\pi} |b_t^{(j)}(re^{i\theta})|^{(1-\beta)/j}\, d\theta$$

$$> C(j, \beta)q^{1-\beta}(1 - r^q)^\beta - \sum_{k=1}^{j} C(k, j, \beta)q^{(1-\beta)}(1 - r^q)^{(1-\beta)/j} > C(j, \beta)p_t.$$

We prove the inequality (3.3) first in the cases $k = j$ and $k = 0$. In the case $k = j$, a minor amendment in the use of the triangle inequality in the preceding argument shows that

$$\int_0^{2\pi} |b_t^{(j)}(re^{i\theta})|^{(1-\beta)/j}\, d\theta < C(j, \beta)p_t. \tag{3.7}$$

In the case $k = 0$, we note that $\sum (1 - |a_n|)^\alpha$, summed over the zeros of B_l, is equal to $\sum_{n=1}^{t-1} p_n 2^{p_n(1-\alpha/\beta)}$. Hence the substitution $\alpha = \beta$ in Theorem 1 yields

$$\int_0^{2\pi} |B_l^{(j)}(re^{i\theta})|^{(1-\beta)/j}\, d\theta < C(j, \beta)\sum_{n=1}^{t-1} p_n < C(j, \beta)p_{t-1}, \tag{3.8}$$

as required.

We can now complete the proof of (3.3) by applying Hölder's inequality in the cases $1 < k < j$. For the left-hand side of (3.3) is bounded above by

$$\left(\int_0^{2\pi} |b_t^{(k)}(re^{i\theta})|^{(1-\beta)/k}\, d\theta\right)^{k/j} \left(\int_0^{2\pi} |B_l^{(j-k)}(re^{i\theta})|^{(1-\beta)/(j-k)}\, d\theta\right)^{(j-k)/j},$$

and we can use (3.7) and (3.8) to estimate the latter two integrals.

In proving (3.4), we consider the logarithmic derivative of B_2 to obtain the formula

$$B_2'(z) = -\sum_{n=t+1}^{\infty} q_n(1 - r_n^{2q_n}) z^{q_n-1} B_{2,n}(z) P_n(z)^2,$$

where $B_{2,n}$ denotes the subproduct of B_2 with the factor b_n omitted, and where

$$P_n(z) = (1 - r_n^{q_n} z^{q_n})^{-1}.$$

Since

$$P_n'(z) = \frac{q_n}{z}(P_n(z)^2 - P_n(z)),$$

an application of the principle of induction establishes the generalisation

$$B_2^{(j)}(z) = \sum_{n=t+1}^{\infty} \sum_{k=0}^{j-1} (1 - r_n^{2q_n}) z^{q_n+k-j} B_{2,n}^{(k)}(z) Q_{j-k}(P_n(z), q_n) \tag{3.9}$$

for $j = 1, 2, 3, \cdots$, where Q_{j-k} is a polynomial of degree $j - k + 1$ in $P_n(z)$ and of degree $j - k$ in q_n with coefficients depending on j and k.

When $j = 1$, the relation (3.9) yields the inequality

$$|B_2'(re^{i\theta})| < \sum_{n=t+1}^{\infty} \frac{q_n r^{q_n-1}}{1 - r^{q_n}} \qquad (0 \le \theta < 2\pi).$$

But since

$$r^{q_n} < q_n r^{q_n} = q_n (1 - 2^{-\gamma p_t})^{q_n} < q_n \exp(-q_n 2^{-\gamma p_t}) = q_n \exp(-p_n 2^{p_n - \gamma p_t})$$

and $p_n \ge p_{t+1} = p_t^2$, it is readily seen that

$$|B_2'(re^{i\theta})| < C(\beta) q_{t+1} \exp(-p_{t+1} 2^{p_t(p_t-\gamma)});$$

the inequality holds not only for B_2, but for all its subproducts $B_{2,n}$ when $n > t$. The application of an inductive argument to (3.9) establishes the general inequality (3.4).

Finally, we need to prove (3.5) and (3.6). To prove the first inequality, we use the relations

$$1 \ge |B_1(re^{i\theta})| \ge \prod_{n=1}^{t-1} \frac{r^{q_n} - r_n^{q_n}}{1 - r^{q_n} r_n^{q_n}} > 1 - \sum_{n=1}^{t-1} \frac{1 - r^{q_n}}{1 - r_n^{q_n}}$$

for $0 \le \theta < 2\pi$, where

$$\sum_{n=1}^{t-1} \frac{1 - r^{q_n}}{1 - r_n^{q_n}} < \sum_{n=1}^{t-1} \frac{(1 - r) r_n^{-q_n}}{1 - r_n} < C(\beta) \sum_{n=1}^{t-1} 2^{\gamma(p_n - p_t)} < C(\beta) 2^{\gamma(p_{t-1} - p_t)}.$$

To prove the second inequality, we use the relations

$$1 \geq |B_2(re^{i\theta})| > 1 - \sum_{n=t+1}^{\infty} \frac{1 - r_n^{q_n}}{1 - r^{q_n}}$$

for $0 < \theta \leq 2\pi$, where

$$r^{q_n} \leq (1 - 2^{-\gamma p_t})^{q_{t+1}} < \exp(-p_{t+1} 2^{p_{t+1} - \gamma p_t})$$

and

$$\sum_{n=t+1}^{\infty} (1 - r_n^{q_n}) < C(\beta)(1 - r_{t+1}^{q_{t+1}}).$$

This completes the proof of Lemma 1.

We can now complete the proof of (i). The Leibnitz formula shows that

$$b^{(M)} = \sum_{j=0}^{M} {}^M C_j (b_t B_1)^{(j)} B_2^{(M-j)}$$

$$= b_t^{(M)} B_1 B_2 + \sum_{k=0}^{M-1} {}^M C_k (b_t^{(k)} B_1^{(M-k)} B_2 + (b_t B_1)^{(k)} B_2^{(M-k)}).$$

The inequalities (3.2), (3.5), and (3.6) show that

$$\int_0^{2\pi} |(b_t^{(M)} B_1 B_2)(re^{i\theta})|^{(1-\beta)/M} d\theta > C(M, \beta) p_t,$$

and the inequality (3.3) shows that

$$\int_0^{2\pi} |((b_t^{(k)} B_1^{(M-k)})(re^{i\theta}))|^{(1-\beta)/M} d\theta \leq C(k, M, \beta) p_t^{k/M} p_{t-1}^{(M-k)/M}$$

for $0 \leq k \leq M$. The inequality (3.3) also leads to the inequality

$$\int_0^{2\pi} |(b_t B_1)^{(k)}(re^{i\theta})|^{(1-\beta)/M} d\theta < \sum_{S=0}^{k} {}^k C_S \int_0^{2\pi} |(b_t^{(S)} B_1^{(k-S)})(re^{i\theta})|^{(1-\beta)/k} d\theta$$

$$< C(k, M, \beta) p_t$$

for $0 < k \leq M$, a result which clearly holds also when $k = 0$. Using this, together with the inequality (3.4), we find that

$$\int_0^{2\pi} |b^{(M)}(re^{i\theta})|^{(1-\beta)/M}\, d\theta$$

$$> C(M, \beta)\, p_t - C(M, \beta)\, p_t\, q_{t+1}^{M!} \exp(-p_{t+1}\, 2^{p_t(p_t-\gamma)}) - C(M, \beta)\, p_t^{(1-1/M)} p_{t-1}^{1/M} .$$

By our choice of p_t, the right-hand side of the last inequality is asymptotically equal to $C(M, \beta)\, p_t$, as $t \to \infty$. This completes the verification of (i).

REFERENCES

1. P. R. Ahern and D. N. Clark, *On inner functions with* H^p-*derivative*. Michigan Math. J. 21 (1974), 115-127.
2. O. Frostman, *Sur les produits de Blaschke*. Kungl. Fysiogr. Sällsk. Lund Förh. 12, no. 15 (1942), 169-182.
3. J. E. Littlewood, *Lectures on the theory of functions*. Oxford Univ. Press, 1944.
4. D. Protas, *Blaschke products with derivative in* H^p *and* B^p. Michigan Math. J. 20 (1973), 393-396.

University College of Swansea, Wales