CERTAIN ALGEBRAIC FUNCTIONS AND EXTREME POINTS OF S

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Let S be the usual set of holomorphic, univalent, normalized ($f(0) = 0$, $f'(0) = 1$) functions on the unit disk $\Delta = \{z \in \mathbb{C}: |z| < 1\}$. In [2] it was shown that if $f \in S$ and the set $\mathbb{C} \setminus f(\Delta)$ contains two points of equal modulus, then f is a convex combination of two other members of S. A simple topological argument leads to the further conclusion that if f is an extreme point of S (see [3, p. 439]), then $\mathbb{C} \setminus f(\Delta)$ is an arc tending to infinity with increasing modulus. (Interesting variations of this result are obtained by W. Hengartner and G. Schober in [4].) In the present note we obtain a generalization of the two-point theorem of [2]. In this generalization we assume that $\mathbb{C} \setminus f(\Delta)$ contains a finite set of points of a certain description, and we conclude that f can be written as a nontrivial convex combination of finitely many members of S. In particular, f is not an extreme point of $\overline{\text{co}}\, S$ (the closure of the convex hull of S). Consequently, the extreme points g of $\overline{\text{co}}\, S$ have the property that the arc $\mathbb{C} \setminus g(\Delta)$ contains no set E of the type described in the theorem below. (The theorem is applicable because, by [3, p. 440], extreme points of $\overline{\text{co}}\, S$ must belong to S.)

THEOREM. *Let* $P(z) = \prod_{j=1}^{n} (z - \alpha_j)$, *where* $n \geq 2$ *and where the* α_j *are distinct complex numbers. Let*

$$Q(z) = \sum_{j=1}^{n} \frac{\lambda_j P(z)}{z - \alpha_j},$$

where the λ_j *are nonzero complex numbers, all having the same argument. Finally, let* E *be the set of complex numbers* w *such that* $P - wQ$ *has a multiple zero. Then* E *consists of* $2n - 2$ *points at most, and any* $f \in S$ *such that* $\mathbb{C} \setminus f(\Delta) \supset E$ *admits an equation of the form* $f = \sum_{j=1}^{n} t_j f_j$ $\left(\sum_{j=1}^{n} t_j = 1,\ t_j > 0,\ f_j \in S,\ f_j \neq f\right)$.

Proof. We begin by noting that Q is a polynomial of degree $n - 1$ and that $Q(\alpha_j) = \lambda_j P'(\alpha_j)$ $(1 \leq j \leq n)$, so that $Q = P'$, in the special case where $\lambda_j = 1$ for each j. In particular, we observe that P and Q have no common zeros. Now suppose $w \in E$ and z is a multiple zero of $P - wQ$. Then $P(z) - wQ(z) = 0$, $Q(z) \neq 0$, and $w = P(z)/Q(z)$. Also, $P'(z) - wQ'(z) = 0$, and hence $(QP' - PQ')(z) = 0$. Since $QP' - PQ'$ is a nontrivial polynomial of degree at most $2n - 2$, there are at most $2n - 2$ such numbers z, and since $w = P(z)/Q(z)$, it follows that there are at most $2n - 2$ such w.

If $|w|$ is sufficiently small, $P - wQ$ has distinct zeros $\phi_j(w)$ $(1 \leq j \leq n)$ (the branches of the algebraic function defined by the equation $P(z) - wQ(z) = 0$). We number these root functions in the natural way so that $\phi_j(0) = \alpha_j$ $(1 \leq j \leq n)$. Each ϕ_j is analytic, and each admits unrestricted analytic continuation in $\mathbb{C} \setminus E$ [1, p. 294]. But $f(\Delta)$ is a simply connected subregion of $\mathbb{C} \setminus E$. Therefore it follows from the monodromy theorem that ϕ_j is analytic and single-valued in $f(\Delta)$ $(1 \leq j \leq n)$.

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Furthermore, each ϕ_j is univalent, for if $w \in f(\Delta)$ and $\phi_j(w) = z$, then $P(z) - wQ(z) = 0$, and therefore $w = P(z)/Q(z)$.

Now, for $w \in f(\Delta)$,

$$\sum_{j=1}^{n} \phi_j(w) = \sum_{j=1}^{n} \alpha_j + w \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \phi_j(0) + \lambda w,$$

where $\lambda = \sum_{j=1}^{n} \lambda_j$. Therefore, solving for w, we are led to the equations

$$w = \sum_{j=1}^{n} \frac{\phi_j'(0)}{\lambda} \psi_j(w), \qquad \psi_j(w) = \frac{\phi_j(w) - \phi_j(0)}{\phi_j'(0)} \qquad (w \in f(\Delta)).$$

For $1 \leq j \leq n$, the function ψ_j is univalent in $f(\Delta)$ and normalized at 0; that is, $\psi_j(0) = 0$ and $\psi_j'(0) = 1$. Our representation of the identity function in $f(\Delta)$ as a linear combination of the ψ_j is actually a convex combination. Indeed, differentiation of the equation $P(\phi_j(w)) - wQ(\phi_j(w)) = 0$ shows that $P'(\phi_j(0))\phi_j'(0) - Q(\phi_j(0)) = 0$, or $\phi_j'(0) = Q(\alpha_j)/P'(\alpha_j) = \lambda_j$. Consequently, $\sum_{j=1}^{n} \phi_j'(0)/\lambda = 1$, and since the numbers λ_j have equal argument, each term $\phi_j'(0)/\lambda$ is real and positive. Finally, the convex combination is nontrivial. In fact, if $\psi_j(w) = w$ identically for some j, then

$$P(\phi_j(0) + \phi_j'(0)w) = P(\phi_j(w)) = wQ(\phi_j(w)) = wQ(\phi_j(0) + \phi_j'(0)w).$$

Hence $P(z) = \dfrac{z - \phi_j(0)}{\phi_j'(0)} Q(z)$ for all z, and this contradicts the fact that P and Q have no zeros in common. Thus the equation

$$f = \sum_{j=1}^{n} \frac{\lambda_j}{\lambda} \psi_j \circ f$$

gives a decomposition of f of the required form.

Remark. To show that this theorem contains the result in [2] described earlier, we choose $n = 2$. Then elementary calculations show that the discriminant of $P - wQ$ is

$$(\lambda_1 + \lambda_2)^2 w^2 + 2(\lambda_1 - \lambda_2)(\alpha_1 - \alpha_2)w + (\alpha_1 - \alpha_2)^2,$$

and that this vanishes for $w = (\alpha_1 - \alpha_2)/(\sqrt{\lambda_2} \pm i\sqrt{\lambda_1})^2$. Thus E consists of two points of equal modulus. Conversely, it is clear that if $|w_1| = |w_2|$ and $w_1 \neq w_2$, then P and Q can be chosen so that $E = \{w_1, w_2\}$. The result of [2] now follows. The author hopes that either he or someone else can extract the information in the theorem corresponding to $n > 2$.

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