A NORM INEQUALITY IN HYPONORMAL OPERATOR THEORY

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1. INTRODUCTION

Recall that a bounded operator T on a Hilbert space $\mathfrak{H}$ is *hyponormal* if

$$T^*T - TT^* = D \geq 0, \tag{1.1}$$

and *completely hyponormal* if, in addition, there is no nontrivial subspace on which T is normal. If $T = H + iJ$ is the Cartesian representation of T, then (1.1) is equivalent to

$$HJ - JH = -iC, \quad \text{where } D = 2C \geq 0. \tag{1.2}$$

It is known that the spectra of H and J are the (real) sets obtained by projecting the spectrum $\sigma(T)$ of T onto the x- and y-axes; see [2, p. 46]. Also, by [3],

$$2\pi \|C\| \leq \operatorname{meas}_2(\sigma(T)). \tag{1.3}$$

Further, if $H = \Re(T)$ has the spectral resolution

$$H = \int t\, dE_t, \tag{1.4}$$

and if T is completely hyponormal, then the spectral family $\{E(\cdot)\}$ is strongly absolutely continuous, that is, $\|E_t f\|^2$ is absolutely continuous in t for each f in $\mathfrak{H}$; see [2, pp. 20, 42].

If α is a Borel set on the real line, then $T_\alpha = E(\alpha)\, T\, E(\alpha)$ is hyponormal, in fact, $T_\alpha^* T_\alpha - T_\alpha T_\alpha^* = E(\alpha)\, D\, E(\alpha) \geq 0$. If $\alpha = \Delta$ is an open interval, and if $E(\alpha) \neq 0$, it follows from the results of [4] that

$$\sigma(T_\Delta) = (\sigma(T) \cap \{z: \Re(z) \in \Delta\})^-, \tag{1.5}$$

where $T_\Delta = E(\Delta)\, T\, E(\Delta)$ is regarded as an operator on $E(\Delta)\mathfrak{H}$. Since $\sigma(E(\Delta)\, J\, E(\Delta))$ is the projection of $\sigma(T_\Delta)$ onto the y-axis, one easily obtains from (1.5) the norm of $E(\Delta)\, J\, E(\Delta)$ (as an operator either on $\mathfrak{H}$ or on $E(\Delta)\mathfrak{H}$) in terms of the spectrum of T in the form

$$\|E(\Delta)\, J\, E(\Delta)\| = \sup\{|\Im(z)|: z \in \sigma(T) \text{ and } \Re(z) \in \Delta\}. \tag{1.6}$$

If F(t) denotes the linear measure of the intersection of $\sigma(T)$ with the line $\Re(z) = t$, so that

$$F(t) = \operatorname{meas}_1[\sigma(T) \cap \{z: \Re(z) = t\}], \tag{1.7}$$

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then, corresponding to (1.3), one has for each open interval Δ the inequality

$$2\pi \, \|E(\Delta)\, C\, E(\Delta)\| \leq \text{meas}_2\,(\sigma(T)) = \int_\Delta F(t)\, dt. \tag{1.8}$$

Next, let α and β be disjoint Borel sets on the real line, so that

$$\alpha \cap \beta = \emptyset. \tag{1.9}$$

In case T is normal, it is well-known that $E(\alpha)\, J\, E(\beta)$ $(= E(\alpha)\, T\, E(\beta)) = 0$. On the other hand, if T is only hyponormal, this relation need not hold. In fact, it is easily shown that if $T = H + iJ$ is a bounded operator and if $E(\alpha)\, J\, E(\beta) = 0$ for all α and β satisfying (1.9), then necessarily $HJ = JH$, so that T must be normal. We shall obtain an estimate for $\|E(\alpha)\, J\, E(\beta)\|$ $(= \|E(\alpha)\, T\, E(\beta)\|)$, whenever α and β satisfy (1.9) and T is hyponormal; the estimate involves the spectrum of T and is, in a certain sense, best possible (see Section 3).

THEOREM. *Let* $T = H + iJ$ *be a hyponormal operator, and suppose that* $H = \mathfrak{R}(T)$ *has the spectral resolution* (1.4). *Let* α *and* β *denote disjoint Borel sets of the real line, so that* (1.9) *holds. Then*

$$\|E(\alpha)\, J\, E(\beta)\| \leq (2\pi)^{-1}\left(\int_\alpha \int_\beta F(x)\, F(y)\, (x - y)^{-2}\, dx\, dy\right)^{1/2}, \tag{1.10}$$

where $F(t)$ *is defined by* (1.7).

2. PROOF OF THE THEOREM

Since each normal part of T can be split off as a component of a direct sum, it is clear that there is no loss of generality in supposing that T is completely hyponormal.

Next, let Δ and δ be two disjoint open (or closed) intervals on the real line for which $d = \text{dist}(\Delta, \delta) > 0$. Then multiply relation (1.2) on the left by $E(\Delta)$ and on the right by $E(\delta)$. If a and b denote the midpoints of Δ and δ, respectively, one obtains the relation

$$\int_\Delta (t - a)\, dE\, J\, E(\delta) - E(\Delta)\, J \int_\delta (t - b)\, dE = -iE(\Delta)\, C\, E(\delta) + (b - a)\, E(\Delta)\, J\, E(\delta),$$

and hence

$$\|E(\Delta)\, J\, E(\delta)\| \leq \|E(\Delta)\, C\, E(\delta)\|/d = \|E(\Delta)\, C^{1/2} C^{1/2}\, E(\delta)\|/d.$$

(See [4, p. 697] for a similar calculation.) In view of (1.8),

$$(2\pi)^{1/2}\|E(\Delta)C^{1/2}\| \leq \left(\int_\Delta F(x)\, dx\right)^{1/2} \quad \text{and} \quad (2\pi)^{1/2}\|C^{1/2}E(\delta)\| \leq \left(\int_\delta F(y)\, dy\right)^{1/2},$$

and consequently,

$$(2.1)\qquad \|E(\Delta)\,J\,E(\delta)\| \le (2\pi)^{-1}\left(\int_\Delta F(x)\,dx \int_\delta F(y)\,dy\right)^{1/2}/d.$$

We shall prove (1.10) first in the special case when

$$(2.2)\qquad e = \operatorname{dist}(\alpha,\beta) > 0.$$

Let $\varepsilon > 0$, and let $\{\Delta_1, \Delta_2, \cdots\}$ be a covering of α by closed intervals that are pairwise essentially disjoint (that is, have at most an end-point in common), and such that $\Delta_n \cap \alpha \neq \emptyset$ and $|\Delta_n| < e/2$ for all n, and $\operatorname{meas}_1\left(\bigcup \Delta_n - \alpha\right) < \varepsilon/2$. If $\{\delta_1, \delta_2, \cdots\}$ is a similar covering for β, then $\{\Delta_j, \delta_k\}$ is a covering of $\alpha \cup \beta$ by essentially disjoint intervals satisfying $d_{jk} = \operatorname{dist}(\Delta_j, \delta_k) > 0$ (for all j, k) and

$$(2.3)\qquad \operatorname{meas}_1\left(\left[\bigcup \Delta_n\right] \cup \left[\bigcup \delta_n\right] - (\alpha \cup \beta)\right) < \varepsilon.$$

If f and g are fixed elements of $\mathfrak{H}$, then, in view of the absolute continuity of $\|E_t f\|^2$ and $\|E_t g\|^2$, it is clear that the intervals $\{\Delta_j, \delta_k\}$ can be chosen so that also

$$(2.4)\qquad \left|\left(E\left(\bigcup \Delta_j\right) J\, E\left(\bigcup \delta_k\right) f,\, g\right) - (E(\alpha)\,J\,E(\beta) f,\, g)\right| < \varepsilon.$$

By (2.1) and the Schwarz inequality, we have the relations

$$\left(E\left(\bigcup \Delta_j\right) J\, E\left(\bigcup \delta_k\right) f,\, g\right) = \left|\sum_j \sum_k (E(\Delta_j)\,J\,E(\delta_k) f,\, g)\right|$$

$$\le \sum_j \sum_k \|E(\Delta_j)\,J\,E(\delta_k)\|\, \|E(\delta_k) f\|\, \|E(\Delta_j) g\|$$

$$\le (2\pi)^{-1}\left(\sum_j \sum_k \left(\int_{\Delta_j} F(x)\,dx \int_{\delta_k} F(y)\,dy\right)/d_{jk}^2\right)^{1/2} \|f\|\, \|g\|.$$

Since ε can be chosen arbitrarily small, relation (1.10), under the restriction (2.2), follows from (2.3) and (2.4) if we let $\sup d_{jk} \to 0$.

Finally, we suppose that α and β satisfy only the condition (1.9). Since the spectrum of $\mathfrak{H}$ is bounded, there is no loss of generality in supposing also that α β are bounded. By a standard result in measure theory, there exist compact sets α_n and β_n such that $\alpha_n \subset \alpha$ and $\beta_n \subset \beta$ (hence $\operatorname{dist}(\alpha_n, \beta_n) > 0$), and such that both $\operatorname{meas}_1(\alpha - \alpha_n) \to 0$ and $\operatorname{meas}_1(\beta - \beta_n) \to 0$ as $n \to \infty$. Clearly, (1.10) holds with α and β replaced by α_n and β_n, respectively. On letting $n \to \infty$ and noting that $E(\alpha - \alpha_n)$ and $E(\beta - \beta_n)$ converge strongly to 0 as $n \to \infty$ (absolute continuity of $E(\cdot)$), one obtains the desired relation (1.10).

3. OPTIMALITY OF THE INEQUALITY

Suppose that $x < y$ and that Δ and δ are open intervals containing x and y, respectively. It follows from (1.10) that for almost all x, y (each variable considered separately),

$$\limsup_{|\Delta|, |\delta| \to 0} \|E(\Delta) J E(\delta)\| / |\Delta|^{1/2} |\delta|^{1/2} \leq (2\pi)^{-1} |x - y|^{-1} (F(x) F(y))^{1/2}. \tag{3.1}$$

We shall show by an example that the equality sign may hold.

To see this, let $\mathfrak{H} = L^2(-1, 1)$, and let the self-adjoint operators H and J be defined by

$$(Hf)(x) = xf(x), \quad (Jf)(x) = (-i\pi)^{-1} \int_{-1}^{1} (y - x)^{-1} f(y)\, dy, \tag{3.2}$$

where the integral operator is a Cauchy principal-value integral. Then $T = H + iJ$ is an irreducible hyponormal operator having the set $[-1, 1] \times [-1, 1]$ as its spectrum; see [1, p. 452]. For convenience, suppose that $-1 < x < y < 1$, where $x \in \Delta$ and $y \in \delta$, and that the open intervals Δ and δ are so small that they lie in $[-1, 1]$ and do not overlap.

If we define $f_\delta = E(\delta) f_\delta$ on $[-1, 1]$ by putting $f_\delta = |\delta|^{-1/2}$ on δ and $f = 0$ otherwise, we see that $\|f_\delta\| = 1$ and

$$(Jf_\delta)(t) \cong (-i\pi)^{-1} d^{-1} |\delta|^{1/2} \quad \text{for } t \in \Delta,$$

where $d = \text{dist}(\Delta, \delta)$ and Δ and δ are small. Thus,

$$\|E(\Delta) J E(\delta) f_\delta\|^2 \cong \pi^{-2} d^{-2} |\Delta| \, |\delta|;$$

more precisely,

$$\|E(\Delta) J E(\delta) f_\delta\|^2 / |\Delta| \, |\delta| \to \pi^{-2} (x - y)^{-2} \quad \text{as } |\Delta|, |\delta| \to 0,$$

and hence

$$\liminf_{|\Delta|, |\delta| \to 0} \|E(\Delta) J E(\delta)\| / |\Delta|^{1/2} |\delta|^{1/2} \geq \pi^{-1} |x - y|^{-1}.$$

But $F(x) = F(y) = 2$, so that equality must hold in (3.1) and also lim sup = lim.

4. AN APPLICATION

It is easy to see from the Theorem that if $\{\Delta_1, \Delta_2, \cdots\}$ is a sequence of intervals for which $|\Delta_n| \to 0$ as $n \to \infty$ and if δ is a fixed interval satisfying the conditions

$$\text{dist}(\Delta_n, \delta) \geq \text{const} > 0, \tag{4.1}$$

then

$$\|E(\Delta_n) J E(\delta)\| \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

On the other hand, it is not clear what happens when (4.1) is replaced by the weaker hypothesis that $\Delta_n \cap \delta = \emptyset$. We now turn to this question and, for definiteness, consider the operators $E((-\infty, 0)) J E((0, 1/n))$ and $E((-1/n, 0)) J E((0, \infty))$.

COROLLARY OF THE THEOREM. *Let* $T = H + iJ$ *be hyponormal, where* H *has the spectral resolution* (1.4) *and* F(t) *is defined by* (1.7). *If*

(4.3) *either* $\int_{-\infty}^{0-} x^{-1} F(x)\,dx < \infty$ *or* $\int_{0+}^{\infty} x^{-1} F(x)\,dx < \infty$,

then both

(4.4) $$\| E((-\infty, 0))\, J\, E((0, 1/n)) \| \to 0 \quad \text{as } n \to \infty$$

and

(4.5) $$\| E((-1/n, 0))\, J\, E((0, \infty)) \| \to 0 \text{ as } n \to \infty .$$

Proof. It will be clear that it is sufficient to prove (4.4) only. Since, by the Theorem,

$$\| E((-\infty, 0))\, J\, E((0, 1/n)) \| \leq (2\pi)^{-1} \left(\int_{-\infty}^{0} \int_{0}^{1/n} F(x)\, F(y)\, (x - y)^{-1}\, dx\, dy \right)^{1/2},$$

the relation (4.4) obviously follows whenever

(4.6) $$M \equiv \int_{-\infty}^{0} \int_{0}^{\infty} F(x)\, F(y)\, (x - y)^{-2}\, dx\, dy < \infty .$$

Suppose that the first relation of (4.3) holds. Since $|F(y)| \leq \text{const} = c$ for all y, we see that

$$M \leq c \int_{-\infty}^{0} F(x) \left[\int_{0}^{\infty} (x - y)^{-2}\, dy \right] dx = c \int_{-\infty}^{0} x^{-1} F(x)\, dx < \infty .$$

A similar argument can be used to establish (4.6) when the second relation of (4.3) holds.

5. REMARKS

Since $F(x) \equiv 0$ when $|x|$ is sufficiently large, we see that a sufficient condition for the validity of (4.4) or (4.5) is that for some constant $k > 0$, $F(x)/x^k \to 0$ as either $x \to 0+$ or $x \to 0-$. However, we do not know whether, for instance, either $F(x) \to 0$ as $x \to 0+$ or $F(x) \to 0$ as $x \to 0-$ (or even both) is sufficient to imply (4.4) or (4.5). Nevertheless, these relations surely cannot hold for an arbitrary hyponormal operator without some smallness restriction on $F(x)$ near $x = 0$. To see this, let $T = H + iJ$, where H and J are defined by (3.2) on $\mathfrak{H} = L^2(-1, 1)$. For $n = 2, 3, \cdots$, let $\delta_n = (1/n, 2/n)$ and $\Delta_n = (-2/n, -1/n)$, and define $f_n = E(\delta_n) f_n$ by putting $f_n = |\delta_n|^{-1/2}$ on δ_n and $f_n = 0$ otherwise. Then $\|f_n\| = 1$, and for $x \in \Delta_n$,

$$E(\Delta_n)(Jf_n)(x) = (-i\pi)^{-1} |\delta_n|^{-1/2} \int_{1/n}^{2/n} (y - x)^{-1}\, dy$$

$$= (-i\pi)^{-1} |\delta_n|^{-1/2} \log[1 + (1 - nx)^{-1}].$$

Thus, $\| E(\Delta_n)\, J\, E(\delta_n)\, f_n \|^2 \geq \pi^{-2} \log^2(4/3) = \text{const} > 0$; hence

$$\|E((-2/n, -1/n))\, J\, E((1/n, 2/n))\| \geq \text{const} > 0 \qquad (n = 2, 3, \cdots),$$

so that, in particular, neither (4.4) nor (4.5) holds.

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