SOME REMARKS ON ERGODICITY AND INVARIANT MEASURES

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Dedicated to Prof. H. Hornich on the occasion of his 70th birthday.

1. INTRODUCTION

The main goal of this paper is to describe a method, applicable to a large class of noninvertible transformations, for proving ergodicity and existence of an invariant measure. We introduce an auxiliary transformation that behaves neatly under some conditions met in applications. This device first appeared in papers by R. Fischer [2] and Schweiger [10], but the approach given in this paper is much more general. For example, we include transformations of the type considered by H. Jager [4] for the very special case of decimal expansions. The model employed is as follows: A pair (M, S), where M is a set and $S: M \to M$ is a map, is called a *fibered system* if there exist a finite or countable set I and a partition $\{B(i) \mid i \in I\}$ of M such that the restriction of S to any $B(i)$ is injective. Let N denote the set of natural numbers, and let X be the set of all functions $f: N \to I$; then the transformation $T: X \to X$, $(Tf)(n) = f(n+1)$ is called the *shift*. The map $\Phi: M \to X$ defined by $\Phi(m)(n) = i$ if $S^{n-1} m \in B(i)$ gives rise to the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & X \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ M & \xrightarrow[\Phi]{} & X \end{array}$$

If Φ is injective (that is, one-to-one), we call Φ *valid* (after W. Parry [5]).

In Section 2, we give the construction of the auxiliary fibered system (M^*, S^*). We show that the validity of Φ implies the validity of the corresponding Φ^*. Section 3 contains a proof that if S^* is ergodic, so is S.

In Section 4, we show that for each invariant measure for S^* one can write down an explicit formula for an invariant set function for S that turns out to be a finite or σ-finite measure in a great number of cases. In Section 5, we discuss the connection between this construction and induced transformations. In particular, we compare the different approaches to Boole's transformation (R. L. Adler and B. Weiss [1], Schweiger [10]).

2. THE CONSTRUCTION

A *cylinder* of rank n is a set

$$B(i_1, \cdots, i_n) = B(i_1) \cap S^{-1} B(i_2) \cap \cdots \cap S^{-n+1} B(i_n).$$

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A cylinder is called *proper* if $S^n B(i_1, \cdots, i_n) = M$. Now let us start with a set $\mathscr{A}$ of cylinders. We define

$$\mathscr{B}_n = \{B(i_1, \cdots, i_n) \mid B(i_1, \cdots, i_s) \notin \mathscr{A} \text{ for } 1 \leq s \leq n-1,\ B(i_1, \cdots, i_n) \in \mathscr{A}\},$$

$$B_n = \bigcup_{E \in \mathscr{B}_n} E, \qquad P = \bigcup_{n=1}^{\infty} B_n.$$

For later use, we also define

$$\mathscr{D}_n = \{B(i_1, \cdots, i_n) \mid B(i_1, \cdots, i_s) \notin \mathscr{A} \text{ for } 1 \leq s \leq n\},$$

$$D_n = \bigcup_{E \in \mathscr{D}_n} E, \qquad W = \bigcap_{n=1}^{\infty} D_n.$$

We now define $V\colon P \to M$ by $Vx = S^n x$ if $x \in B_n$. Note that $W = M \setminus P$. The pair (M^*, S^*), where $M^* = M \setminus \bigcup_{j=0}^{\infty} V^{-j}W$ and S^* is the restriction of V to M^*, is a fibered system with index set

$$J = \bigcup_{n=1}^{\infty} \{(i_1, \cdots, i_n) \in I^n \mid B(i_1, \cdots, i_n) \in \mathscr{B}_n\}.$$

We denote a cylinder with respect to S^* by $B^*(j_1, \cdots, j_m)$. Clearly, the interesting case will be $M = M^*$ "almost".

THEOREM 1. *If* $\Phi\colon M \to X$ *is valid, then the corresponding map* $\Phi^*\colon M^* \to X^*$ *is valid.*

Proof. Observe that Φ is valid if and only if the intersection

$$\bigcap_{n=1}^{\infty} B(i_1, \cdots, i_n)$$

of each sequence of nested cylinders contains at most one point. Each digit j of the system (M^*, S^*) is a block $(i_1, \cdots, i_s) \in I^s$ for some s.

We list two general examples: Fix a digit $r \in I$ and take

$$\mathscr{A}(r) = \{B(i_1, \cdots, i_s) \mid i_s \neq r\}.$$

The special case of decimal expansions (that is, the map $x \to 10x \bmod 1$) is considered in a paper by Jager [4]. Another useful type is obtained if we take $\mathscr{A}$ to be the class of proper cylinders (Fischer [2]) or a suitable subclass of it (Schweiger [10]).

3. ERGODICITY

Now let us assume that M has the structure of a measure space $(M, \mathscr{F}, \lambda)$, where $\mathscr{F}$ is a σ-algebra and λ is a finite or σ-finite measure. We always assume that $E \in \mathscr{F}$ implies $SE \in \mathscr{F}$ and $S^{-1}E \in \mathscr{F}$. Taking intersections, we arrive at a measure space $(M^*, \mathscr{F}^*, \lambda)$, where we write λ instead of λ^*.

We shall use the formulas

$$V^{-1} E = \dot{\bigcup_{n=1}^{\infty}} (B_n \cap S^{-n} E), \qquad VE = \dot{\bigcup_{n=1}^{\infty}} S^n(B_n \cap E)$$

(in the latter, $E \subseteq P$ is assumed). These formulas imply immediately the following result.

THEOREM 2. *Let* $\mathscr{A}$ *be a class of cylinders such that* $\lambda(W) = 0$. *If* $\lambda(E) = 0$ *implies* $\lambda(S^{-1} E) = 0$ *and* $\lambda(SE) = 0$, *then* $M = M^*$ mod 0 *and* $\lambda(E) = 0$ *also implies* $\lambda((S^*)^{-1} E) = 0$ *and* $\lambda(S^* E) = 0$. *Furthermore, if* S^* *is ergodic, so is* S.

In the rest of the paper, we make the assumptions of Theorem 2.

Note that $\lim_{n \to \infty} \lambda(D_n) = 0$ is a sufficient condition (also necessary, if λ is finite) to ensure $\lambda(W) = 0$. If $\mathscr{A}$ is a class of proper cylinders, the conditions of Theorem 2 ensure that all cylinders of the system (M^*, S^*) are proper.

We say that a class of cylinders $\mathscr{A}$ has *weak playback* if $B(i_1, \cdots, i_n) \in \mathscr{A}$ and $B(k_1, \cdots, k_m) \in \mathscr{A}$ imply $B(i_1, \cdots, i_n, k_1, \cdots, k_m) \in \mathscr{A}$. Let $\mathscr{B}^{(s)}$ denote the σ-algebra generated by all cylinders of rank at most s, and let $\mathscr{B} = \bigvee_{s=1}^{\infty} \mathscr{B}^{(s)}$ denote their limit.

THEOREM 3 (see Fischer [2, Lemma 1]). *Let* $\mathscr{A}$ *be a class of cylinders with weak playback and such that* $\lim_{n \to \infty} \lambda(D_n) = 0$. *Then* $\mathscr{B}^* = \mathscr{B}$.

Proof. We show that every cylinder E of the system (M, S) is a disjoint union of cylinders from $\mathscr{A}$. The equality $D_n = B_{n+1} \cup D_{n+1}$ shows

$$D_n = \dot{\bigcup_{k=1}^{\infty}} B_{n+k} \text{ mod } 0.$$

Let $E = B(i_1, \cdots, i_t)$. If $E \in \mathscr{D}_t$, then

$$E = \dot{\bigcup_{k=1}^{\infty}} B_{t+k} \cap E \text{ mod } 0.$$

If $E \notin \mathscr{D}_t$, then there is a maximal $r \in [1, t]$ such that $B(i_1, \cdots, i_r) \in \mathscr{A}$. The weak-playback property implies that $F = B(i_{r+1}, \cdots, i_t) \in \mathscr{D}_{t-r}$ (otherwise $B(i_{r+1}, \cdots, i_s) \in \mathscr{A}$ for some $s < t$, and then $B(i_1, \cdots, i_r, i_{r+1}, \cdots, i_s) \in \mathscr{A}$, contradicting the choice of r). The weak-playback property shows that

$$E = B(i_1, \cdots, i_r) \cap S^{-r}\left(\dot{\bigcup_{k=1}^{\infty}} B_{t-r+k} \cap F\right) \text{ mod } 0$$

is a disjoint-union representation by cylinders taken from $\mathscr{A}$.

Most results on continued-fraction-like algorithms deal with proper cylinders (for theorems and references, see A. Rényi [6], Schweiger [8], S. M. Rudolfer [7]). For various cases, it can be shown that $\lim_{n \to \infty} \lambda(D_n) = 0$ (see Schweiger [9] for Jacobi's algorithm, Fischer [3] for references on matrix algorithms, and M. S. Waterman [11] for sufficient conditions). Theorem 3 tells us that we can use conditional expectations and martingale theorems with respect to $\mathscr{B}^*$ instead of $\mathscr{B}$.

4. INVARIANT MEASURE

THEOREM 4. *Let* $\mathscr{A}$ *be a class of cylinders with weak playback and such that moreover* $\sum_{n=1}^{\infty} \lambda(D_n) < \infty$. *If* S^* *admits a finite invariant measure* ν *such that* $\nu(A) \leq c\lambda(A)$ *with a constant* $c > 0$, *then* S *admits a finite invariant measure* $\mu \sim \nu$.

Proof. We put $D_0 = \bigcup_{n=1}^{\infty} B_n = M \bmod 0$. We claim that

$$\mu(A) = \sum_{n=0}^{\infty} \nu(S^{-n} A \cap D_n)$$

is a finite invariant measure with respect to S. We note that $D_n = D_{n+1} \cup B_{n+1}$ and $\nu(S^{-n-1} A \cap D_n) = \nu(S^{-n-1} A \cap D_{n+1}) + \nu(S^{-n-1} A \cap B_{n+1})$. Now

$$\mu(S^{-1} A) = \sum_{n=0}^{\infty} \nu(S^{-n-1} A \cap D_n) = \sum_{n=0}^{\infty} \nu(S^{-n-1} A \cap D_{n+1}) + \sum_{n=0}^{\infty} \nu(S^{-n-1} A \cap B_{n+1})$$

$$= \sum_{n=1}^{\infty} \nu(S^{-n} A \cap D_n) + \sum_{n=1}^{\infty} \nu(S^{-n} A \cap B_n) = \sum_{n=1}^{\infty} \nu(S^{-n} A \cap D_n) + \nu(S^*)^{-1} A),$$

but $\nu((S^*)^{-1} A) = \nu(A)$, by assumption. Since S is nonsingular, $\nu(A) = 0$ is equivalent with $\mu(A) = 0$. The estimate

$$\mu(A) \leq \sum_{n=1}^{\infty} \nu(D_n) + \nu(A)$$

shows that μ is finite.

We say that a class of cylinders $\mathscr{A}$ has *strong playback* if $B(i_1, \cdots, i_n) \neq \emptyset$ and $B(k_1, \cdots, k_m) \in \mathscr{A}$ imply $B(i_1, \cdots, i_n, k_1, \cdots, k_m) \in \mathscr{A}$.

THEOREM 5. *Let* $\mathscr{A}$ *be a class of cylinders with strong playback and such that* $\lim_{n \to \infty} \lambda(D_n) = 0$. *If* S^* *admits a finite invariant measure* ν, *then* S *admits a* σ*-finite* (*or infinite*) *invariant measure* $\mu \sim \nu$.

Proof. The formula

$$\mu(A) = \sum_{n=0}^{\infty} \nu(S^{-n} A \cap D_n)$$

is again an invariant set function $\mu \sim \nu$. Since $\bigcup_{m=1}^{\infty} B_m = M \bmod 0$, we shall prove that $\mu(A) < \infty$ for every cylinder $A \in \mathscr{B}_m$. Observe that for each $B \in \mathscr{D}_n$ the strong-playback property implies $S^{-1} A \cap B \in \mathscr{A}$ and therefore $S^{-n} A \cap D_n \leq D_n \setminus D_{n+m}$. This shows that

$$\mu(A) \leq \sum_{n=0}^{\infty} \nu(D_n \setminus D_{n+m}) \leq \sum_{n=0}^{m-1} \nu(D_n) < \infty .$$

Again, if $\mathscr{A}$ consists of proper cylinders only, then all cylinders of the fibered system (M^*, S^*) are proper. It is known that in this case the existence of invariant measures is more likely. We remark that, taking densities, one obtains the Parry-Fischer formula for the special cases considered there (see Fischer [2]).

5. INDUCED TRANSFORMATIONS

Suppose $A \subseteq M$ has the property $\bigcup_{n=1}^{\infty} S^{-n} A = M \bmod 0$. For almost every $x \in M$, there is a minimal $n(x) \in N$ such that $S^{n(x)} x \in A$. The map $H: A \to A$, $Ha = S^{n(a)} a$ is called the *induced transformation.* If we take intersections and the restriction of the measure, the measure space $(M, \mathscr{F}, \lambda)$ inherits a measure-space structure $(A, \mathscr{F}', \lambda)$, where we write λ instead of λ'. It is known (Adler and Weiss [1]) that S is ergodic if and only if H is ergodic. If $\mu \ll \lambda$ is a measure preserved by S and $\mu(A) < \infty$, then H preserves (the restriction of) μ.

We define $A_k = \{x \in A \mid n(x) = k\}$. Now we consider a class $\mathscr{A}$ of cylinders such that $\lim_{n \to \infty} \lambda(D_n) = 0$, $\mathscr{A}$ has strong playback, and in addition $B_{n+1} = D_n \cap S^{-n} B_1$ for all $n \in N$. This condition is a kind of decoding condition. The class $\mathscr{A}(r)$ mentioned in Section 2 has this property.

THEOREM 6. *Let $\mathscr{A}$ satisfy the conditions mentioned before. Furthermore, let S^* admit a finite measure and let ν admit the σ-finite (or finite) measure μ, the two measures being related by*

$$\mu(E) = \sum_{n=0}^{\infty} \nu(S^{-n} E \cap D_n).$$

If we form the induced transformation of S *with respect to* $A = B_1$, *then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{H} & A \\ {\scriptstyle S_A}\downarrow & & \downarrow{\scriptstyle S_A} \\ M & \xrightarrow[S^*]{} & M \end{array}$$

is commutative. $S_A: A \to M$ *preserves measure, that is, the inversion formula*

$$\mu(S^{-1} F \cap A) = \nu(F)$$

holds for measurable $F \subseteq M$.

Proof. Our conditions imply $\mu(A) < \infty$ (see the proof of Theorem 5). We claim that

$$A_k = S^{-1} B_k \cap A.$$

Let $x \in S^{-1} B_k \cap A = S^{-1}(D_{k-1} \cap S^{-k+1} B_1) \cap B_1$. Clearly, $S^k x \in A$. Choose j so that $1 \leq j < k$; then $S^j x \in S^{j-1} D_{k-1}$. Let $B(i_1, \cdots, i_{k-1}) \in \mathscr{D}_{k-1}$; then $S^{j-1} B(i_1, \cdots, i_{k-1}) \leq B(i_j, \cdots, i_{k-1})$. If $B(i_j, \cdots, i_{k-1}) \cap B_1 \neq \emptyset$, then $B(i_j) \in \mathscr{B}_1$ and therefore $B(i_1, \cdots, i_j) \in \mathscr{B}_j$, which contradicts $B(i_1, \cdots, i_{k-1}) \in \mathscr{D}_{k-1}$. Therefore $S^j x \notin A$.

Now

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (S^{-1} B_k \cap A) = S^{-1} \bigcup_{k=1}^{\infty} B_k \cap A = A,$$

which shows that $A = B_1 \subseteq \bigcup_{n=1}^{\infty} S^{-n} A$. By one of our assumptions, $B_{n+1} \subseteq S^{-n} A$, and therefore

$$M = \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} S^{-n} A \bmod 0.$$

If $x \in A$, then for some k we have the relation $x \in A_k$. Hence

$$S_A Hx = S^{k+1} x = S^* S_A x$$

(since $Sx \in B_k$). A calculation shows that

$$\mu(S^{-1} F \cap A) = \sum_{j=0}^{\infty} \nu(S^{-j-1} F \cap S^{-j} A \cap D_j)$$

$$= \sum_{i=1}^{\infty} \nu(S^{-i} F \cap B_i) = \nu((S^*)^{-1} F) = \nu(F).$$

Remark. Let us suppose $I = \{a, b\}$ and $B(b)$ is proper. If we take

$$\mathscr{A} = \mathscr{A}(a) = \{B(i_1, \cdots, i_n) \mid i_n \neq a\},$$

then $A = B(b)$ and $S_A: A \to M$ is an isomorphism.

In [1], Adler and Weiss proved that $S: R \to R$, $Sx = x - 1/x$ is ergodic with respect to Lebesgue measure λ (which is actually an invariant measure). They showed that the induced transformation H on $A = [-1, 1]$ is ergodic. In [10], the fibered system (M, T), $M = [0, 1]$, $Tx = (1/\pi)\arctan((\tan 2\pi x)/2)$ with the partition $B(0) = [0, 1/2]$, $B(1) = [1/2, 1]$ was considered.

It was shown that the map T^* constructed with respect to the class

$$\mathscr{A} = \{B(i_1, \cdots, i_{s-1}, i_s) \mid (i_{s-1}, i_s) = (0, 1) \text{ or } (1, 0)\}$$

is ergodic with respect to Lebesgue measure λ on $[0, 1]$ and admits a finite invariant measure ν. This proved that T itself is ergodic with respect to λ and admits a σ-finite invariant measure μ. Since the map ψ: $[0, 1] \to R$, $\psi x = \tan(x\pi - \pi/2)$ establishes an isomorphism between the systems $([0, 1], T, \mu)$ and (R, S, λ), this was another proof of the ergodicity of Boole's transformation. But if we introduce a new partition in $[0, 1]$, namely

$$B(a) = B(0, 0) \cup B(1, 1), \qquad B(b) = B(0, 1) \cup B(1, 0),$$

then $\mathscr{A} = \mathscr{A}(a)$ and $\psi B(b) = [-1, 1]$. This shows that the systems $([0, 1], T^*, \nu)$ and (A, H, λ) are isomorphic. The inversion formula of Theorem 6 can be used to calculate ν explicitly for this example (and examples (1) and (2) in Schweiger [10]).

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