DISTRIBUTIVE NOETHER LATTICES

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1. INTRODUCTION

R. P. Dilworth [4] introduced the concept of a Noether lattice as an abstraction of the concept of the lattice of ideals of a Noetherian ring. In [2], K. P. Bogart showed that a distributive regular local Noether lattice of dimension n is isomorphic to the sublattice RL_n of ideals of $F[X_1, \cdots, X_n]$, where F is a field, generated by the principal ideals $(X_1), \cdots, (X_n)$ under multiplication and joins. In [3], Bogart further showed that each distributive local Noether lattice is a certain quotient of RL_n. E. W. Johnson and J. P. Lediaev [7] characterized the distributive Noether lattices that can be represented as the lattice of ideals of a Noetherian ring.

By a semigroup, we shall mean a commutative semigroup with 0 and 1, written multiplicatively. In Section 2 we show that the lattice of ideals of a semigroup is a quasi-local distributive multiplicative lattice, and we give a necessary and sufficient condition for a multiplicative lattice to be the lattice of ideals of a semigroup. As a corollary, it follows that a distributive local Noether lattice is the lattice of ideals of a certain type of semigroup. In Section 3 we show that every distributive local Noether lattice can be embedded in the lattice of ideals of a Noetherian ring. In Section 4 we consider distributive regular Noether lattices. Two characterizations of distributive regular Noether lattices are given. An interesting result proved in this section is that if in a principally generated distributive multiplicative lattice two primes are not comaximal, then their join is again prime.

2. SEMIGROUPS

Let L be a multiplicative lattice. An element M of L is *meet (join-) principal* if $AM \wedge B = (A \wedge (B:M))M$ (if $(AM \vee B:M) = A \vee (B:M)$) for all A and B in L. An element M of L is *weak-meet (weak-join) principal* if

$$M \wedge B = (B:M)M \qquad ((BM:M) = B \vee (0:M))$$

for all B in L. A *principal element* is an element that is both meet-principal and join-principal. We say that a multiplicative lattice is *quasi-local* if it has a unique maximal element.

Let S be a commutative semigroup with 0 and 1, written multiplicatively. The set-theoretic union or intersection of each set of ideals is again an ideal. Thus L(S), the lattice of ideals of S, is easily seen to be an infinitely distributive multiplicative lattice. (Recall that a lattice L is said to be infinitely distributive if $A \wedge \left(\bigvee_\alpha B_\alpha\right) = \bigvee_\alpha (A \wedge B_\alpha)$ for all $A \in L$ and $\{B_\alpha\} \subseteq L$, where $\{\alpha\}$ is an arbitrary indexing set.) The set of nonunits of S forms the unique maximal ideal of S, so that L(S) is quasi-local. Each principal ideal (x) of S is meet-principal in L(S); however, it need not be weak-join-principal. For example, if $S = \{0, 1, x, y\}$

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with multiplication $x = x^2$, $y = y^2$, and $xy = 0 = yx$, then (x) and (y) are not weak-join-principal in $L(S)$. Note that (x, y) is principal in $L(S)$, but is not join-irreducible.

A principal ideal (x) of S is (weak-) join-principal if and only if for all $b, c \in S$, $xb = xc \neq 0$ implies $(b) = (c)$. Thus $L(S)$ is principally generated if and only if for every $x \in S$, $xb = xc \neq 0$ implies $b = \lambda c$ for some unit $\lambda \in S$. A semigroup satisfying this condition will be called a *weak-cancellation semigroup*. For a weak-cancellation Noetherian semigroup S, $L(S)$ is a distributive local Noether lattice. The converse is also true, and it will follow from the next theorem.

THEOREM 1. *Let* L *be a multiplicative lattice.* L *is isomorphic to the lattice of ideals of a semigroup if and only if*

(1) L *is infinitely distributive,*

(2) L *is quasi-local, and*

(3) *there exists a set* S *of weak-meet-principal elements of* L *that generates* L *under joins, and is closed under products, and whose elements are join-irreducible.*

Proof. Clearly, the lattice of ideals of a semigroup satisfies (1), (2), and (3). Now (2) and (3) imply that S is a semigroup (with 0 and 1). Let $L(S)$ be the lattice of ideals of S; we show that $L(S)$ is isomorphic to L via the map $\theta: L(S) \to L$ given by $\theta(J) = \bigvee\{x \in S \mid x \in J\}$ for each ideal J of S. Clearly, θ is well-defined, preserves order, joins, and products, and is surjective. Thus it remains to show that for two ideals J and K in S, the inequality $\theta(J) \leq \theta(K)$ implies $J \subseteq K$. For $x_0 \in J$, $x_0 \leq \theta(J) \leq \theta(K) = \bigvee\{x \in S \mid x \in K\}$, so that by the distributive law

$$x_0 = x_0 \wedge \left(\bigvee\{x \in S \mid x \in K\}\right) = \bigvee\{x_0 \wedge x \mid x \in K\}.$$

Since each $x \in K$ is weak-meet-principal, $x_0 = \bigvee\{(x_0 : x)x \mid x \in K\}$. Now each $(x_0 : x)$ is a join of elements of S; also, S is closed under products, and x_0 is join-irreducible. Thus x_0 must be a multiple of some $x \in K$, so that $x_0 \in (x) \subseteq K$.

In a local Noether lattice, a weak-meet-principal element is principal and join-irreducible [2]. The following theorem now follows immediately from Theorem 1.

THEOREM 2. *A distributive local Noether lattice is the lattice of ideals of a weak-concellation Noetherian semigroup.*

Suppose L is a distributive local Noether lattice. Then L is isomorphic to $L(S)$, where S is a weak-cancellation Noetherian semigroup. If we define $a \sim b$ whenever $(a) = (b)$, then $S^* = S/\sim$ is a Noetherian semigroup with the property that $xa = xb \neq 0$ implies $a = b$; moreover, $L(S)$ and $L(S^*)$ are isomorphic.

3. AN EMBEDDING THEOREM

We say that a Noether lattice L is *embeddable* if there exist a Noetherian ring R and a product-preserving lattice monomorphism $L \to L(R)$ that sends principal elements to principal elements and 0 to 0.

THEOREM 3. *A distributive local Noether lattice is embeddable.*

Proof. Let L be a distributive local Noether lattice. By Theorem 2, $L \cong L(S)$, where S is a weak-cancellation Noetherian semigroup whose only unit is 1. Let

$\{x_1, \cdots, x_n\}$ be a minimal basis for the maximal ideal of S. Let Z_0 be the additive semigroup of nonnegative integers, and let $T = Z_0 \times \cdots \times Z_0$ (n times). We define the congruence $\sim$ on T by the rule that $(a_1, \cdots, a_n) \sim (b_1, \cdots, b_n)$ if

$$x_1^{a_1} \cdots x_n^{a_n} = x_1^{b_1} \cdots x_n^{b_n}.$$

Let $\mathscr{S}$ be the semigroup $T/\sim$, let F be a field, and let $F[X, \mathscr{S}]$ be the semigroup ring over $\mathscr{S}$ with coefficients in F. The basis elements of $F[X, \mathscr{S}]$ will be denoted by $X\overline{(a_1, \cdots, a_n)}$, where $\overline{(a_1, \cdots, a_n)} \in \mathscr{S}$. Since $F[X, \mathscr{S}]$ is a homomorphic image of the polynomial ring $F[X_1, \cdots, X_n]$, it is Noetherian. The map $\theta: S \to F[X, \mathscr{S}]$ determined by

$$x_1^{a_1} \cdots x_n^{a_n} \mapsto X\overline{(a_1, \cdots, a_n)}$$

induces a map $\theta: L(S) \to L(F[X, \mathscr{S}])$. (The congruence $\sim$ ensures that both maps are well-defined.) Clearly, θ preserves joins, products, and order, and it takes 0 to 0. Since principal elements in L(S) are join-irreducible, θ takes principal elements to principal ideals.

Next we show that θ is injective. Let $J, K \in L(S)$, and suppose $\theta(J) = \theta(K)$. For $x_1^{a_1} \cdots x_n^{a_n} \in J$, it suffices to show $x_1^{a_1} \cdots x_n^{a_n} \in K$. Now

$$(X\overline{(a_1, \cdots, a_n)}) = (\theta(x_1^{a_1} \cdots x_n^{a_n})) \subseteq \theta(K),$$

so that $X\overline{(a_1, \cdots, a_n)} = f_1 j_1 + \cdots + f_m j_m$, where $f_1, \cdots, f_m \in F[X, \mathscr{S}]$ and $j_1, \cdots, j_m$ are homogeneous elements in $\theta(K)$ with $j_i = \theta(y_i)$ for some $y_i \in K$. Writing each f_i as a sum of homogeneous elements and collecting terms, we obtain the relation $X\overline{(a_1, \cdots, a_n)} = fj$, where $f \in F[X, \mathscr{S}]$ is homogeneous and $j \in \theta(K)$ is homogeneous with $j = \theta(y)$ for some $y \in K$. Thus $x_1^{a_1} \cdots x_n^{a_n} \in (y) \subseteq K$.

It remains to show that θ preserves meets. This follows easily from the fact that the images of ideals of S are homogeneous in $F[X, \mathscr{S}]$.

Bogart [2] has defined a Noether lattice embedding as an embedding that furthermore preserves prime and primary elements. Our embedding need not be a Noether lattice embedding. However, if $\mathscr{S}$ is a torsionless cancellation semigroup, then a homogeneous ideal of $F[X, \mathscr{S}]$ is prime (primary) if and only if it is prime (primary) with respect to homogeneous elements [8, p. 124]. Thus, if $\mathscr{S}$ is a torsionless cancellation semigroup, then the embedding is actually a Noether lattice embedding. Applying this construction to RL_n, we find that $\mathscr{S} = Z_0 \times \cdots \times Z_0$ (n times), and hence we obtain Bogart's embedding into $F[X, \mathscr{S}] = F[X_1, \cdots, X_n]$.

In [7], E. W. Johnson and J. P. Lediaev prove that a distributive Noether lattice is representable as the lattice of ideals of a Noetherian ring if and only if it satisfies the weak-union condition. (Recall that a lattice L satisfies the *weak-union condition* if for each set of three elements A, B, and C of L with $A \not\leq B$ and $A \not\leq C$, there exists a principal element $E \leq A$ with $E \not\leq B$ and $E \not\leq C$.) A multiplicative lattice is called an r-*lattice* if it is modular, principally generated, and compactly generated, and if its greatest element is compact [1]. Observing that a lattice-ordered abelian group is the group of divisibility of a Bézout domain, we can show that a distributive r-lattice domain L can be represented as the lattice of ideals of a ring (necessarily a Prüfer domain) if and only if L satisfies the weak-union condition.

4. REGULAR DISTRIBUTIVE NOETHER LATTICES

We begin with a short proof that a distributive regular local Noether lattice of dimension n is isomorphic to RL_n [2]. First we remark that RL_n is isomorphic to the lattice of ideals of S_n, the free semigroup on n elements.

THEOREM 4. *Every distributive regular local Noether lattice* L *of dimension* n *is isomorphic to* $L(S_n)$, *the lattice of ideals of the free semigroup on* n *elements.*

Proof. By Theorem 2, L is isomorphic to L(S), the lattice of ideals of S, where S is a Noetherian weak-cancellation semigroup whose only unit is 1. Now S is a homomorphic image of S_n, say $\theta: S_n \to S$ with $X_i \to y_i$, where $X_1, \cdots, X_n$ ($y_1, \cdots, y_n$) is a minimal basis for S_n (for S). Since $\dim S_n = \dim S$, 0 is prime in S; thus S is a cancellation semigroup. To show θ is an isomorphism, it suffices to show each (y_i) is prime. But $S/(y_i)$ is again regular and hence a domain; therefore (y_i) must be prime.

We mention another interpretation of RL_n. Let R be a local ring, and let $x_1, \cdots, x_n$ be an R-sequence of length n. The collection of all ideals generated by monomials in $x_1, \cdots, x_n$ is a distributive regular local Noether lattice of dimension n [6], [9]. Theorem 8 is a global generalization.

As for rings, we define a Noether lattice to be regular if L_M is a regular local Noether lattice for every maximal element M of L. A regular Noether lattice is easily seen to be a finite direct product of regular Noether lattice domains. In particular, a distributive regular Noether lattice is a finite direct product of distributive regular Noether lattice domains. A principally generated multiplicative lattice domain is called a UFD if every principal element is a product of principal primes. A distributive Noether lattice domain is regular if and only if it is a UFD. Further, any localization of a distributive regular Noether lattice is still regular.

The next theorem, while interesting in its own right, will be used to give our first characterization of distributive regular Noether lattice domains.

THEOREM 5. *Suppose* L *is a principally generated distributive multiplicative lattice. For any two primes* P *and* Q *of* L, *either* $P \vee Q$ *is prime or* $P \vee Q = I$, *where* I *is the greatest element of* L.

Proof. Let A and B be principal elements, with $AB \leq P \vee Q$. By the distributive law, $AB = (AB \wedge P) \vee (AB \wedge Q)$. Because AB is principal, there exist C and D in L such that $AB \wedge P = ABC$ and $AB \wedge Q = ABD$. Hence

$$AB = ABC \vee ABD = AB(C \vee D).$$

Since AB is principal, $I = C \vee D \vee (0 : AB)$. We can assume $AB \not\leq P$ and $AB \not\leq Q$; therefore $ABC \leq P$ implies $C \leq P$, and $ABD \leq Q$ implies $D \leq Q$. Hence $I = P \vee Q \vee (0 : AB) = P \vee Q$, since $AB(0 : AB) = 0 \leq P$ implies $(0 : AB) \leq P$.

Remark. If in a principally generated distributive multiplicative lattice A is P-primary, B is Q-primary, and A and B are not comaximal, then $A \vee B$ is $P \vee Q$-primary.

THEOREM 6. *Let* L *be a regular distributive Noether lattice domain, and let* P *be a nonzero prime in* L. *Then* rank P = r *if and only if there exist* r *distinct nonzero principal primes* $P_1, \cdots, P_r$ *whose join is* P. *Furthermore, these principal primes are uniquely determined.*

Proof. The proof will be by induction on $r = \text{rank } P$. The case $r = 1$ needs no further proof, since L is a UFD. Suppose the result is proved for all primes of rank less than r, and let $Q < P$ be a prime of rank $r - 1$. By induction, $Q = P_1 \vee \cdots \vee P_{r-1}$, where $P_1, \cdots, P_{r-1}$ are distinct nonzero principal primes. Now $Q < P$ implies there exists a principal element A with $Q < Q \vee A \leq P$. Since L is a UFD, we may write A as a product $Q_1 \cdots Q_n$ of principal primes. Since P is prime, say $Q_1 \leq P$, then

$$Q < Q \vee A \leq Q \vee Q_1 = P_1 \vee \cdots \vee P_{r-1} \vee Q_1 \leq P,$$

and $P_1 \vee \cdots \vee P_{r-1} \vee Q_1$ is prime. Therefore $P = P_1 \vee \cdots \vee P_{r-1} \vee Q_1$.

Conversely, suppose $P = P_1 \vee \cdots \vee P_r$, where $P_1, \cdots, P_r$ are distinct nonzero principal primes. Now L_P is a regular distributive local Noether lattice, and $P_{1P}, \cdots, P_{rP}$ are distinct nonzero principal primes in L_P. Hence

$$\text{rank } P = \text{rank } P_P = r.$$

We are now in a position to characterize distributive regular Noether lattices. For a multiplicative lattice L, we define $\text{Spec}(L) = \{P \in L \mid P \text{ is prime}\}$. Clearly, two regular distributive local Noether lattices are isomorphic if and only if their spectra are isomorphic as posets. We extend this result to arbitrary distributive regular Noether lattice domains.

THEOREM 7. *Two distributive regular Noether lattice domains* L *and* L' *are isomorphic if and only if* Spec (L) *and* Spec (L') *are isomorphic as posets.*

Proof. Let $\theta: \text{Spec}(L) \to \text{Spec}(L')$ be an isomorphism of posets. If P is a prime of rank one (that is, a principal prime) in L, then $\theta(P)$ is a principal prime of rank one in L'. We first extend θ to principal elements of L. Since L is a UFD, each principal element X has a unique decomposition into a product of principal primes, say $X = P_1 \cdots P_n$. We define $\theta(X) = \theta(P_1) \cdots \theta(P_n)$. This map is well-defined and injective, and it preserves products of principal elements and maps onto the set of principal elements of L'. We extend θ to $L \to L'$ by linearity: if $X = X_1 \vee \cdots \vee X_n$, where $X_1, \cdots, X_n \in L$ are principal, then we define

$$\theta(X) = \theta(X_1) \vee \cdots \vee \theta(X_n)$$

in L'. First, we must show that $\theta: L \to L'$ is well-defined. Suppose

$$X_1 \vee \cdots \vee X_n = Y_1 \vee \cdots \vee Y_m$$

are two representations of $A \in L$ as joins of principal elements. We must show that $\theta(X_1) \vee \cdots \vee \theta(X_n) = \theta(Y_1) \vee \cdots \vee \theta(Y_m)$. Now, for each maximal element M of L, $\theta_M: \text{Spec}(L_M) \to \text{Spec}(L'_{\theta(M)})$ is an isomorphism of posets, and hence it extends uniquely to an isomorphism $\theta_M: L_M \to L'_{\theta(M)}$. Thus

$$(\theta(X_1) \vee \cdots \vee \theta(X_n))_M = (\theta(Y_1) \vee \cdots \vee \theta(Y_n))_M$$

for every maximal element M of L. Hence

$$\theta(X_1) \vee \cdots \vee \theta(X_n) = \theta(Y_1) \vee \cdots \vee \theta(Y_m).$$

A similar proof shows that θ is injective. The remaining details are easily verified.

Suppose L is a distributive regular Noether lattice domain, and let K be the set of nonzero principal primes of L. Let FP(K) be the set of all finite subsets of K. We may embed Spec(L) into FP(K) by sending 0 to the null set, and $P = P_1 \vee \cdots \vee P_n$ to $\{P_1, \cdots, P_n\}$, where $P_1 \vee \cdots \vee P_n$ is the unique representation of P as a join of principal primes (Theorem 6). As a subset of FP(K), X = Spec(L) satisfies the conditions

(1) the ascending chain condition (ACC),

(2) $Z \subseteq Y \in X$ implies $Z \in X$, and

(3) $\{P\} \in X$ for every nonzero principal prime P.

Conversely, suppose K is any set, and let $X \subseteq FP(K)$, the set of all finite subsets of K. Further, suppose X satisfies the conditions (1), (2), and (3). Then there exist a (unique) distributive regular Noether lattice domain L, a bijection θ from the set of nonzero principal primes of L onto K, and an isomorphism of posets $\hat{\theta}: \text{Spec}(L) \to X$ such that for each $P \in \text{Spec}(L)$, $\hat{\theta}(P) = \{\theta(P_1), \cdots, \theta(P_n)\}$, where $P = P_1 \vee \cdots \vee P_n$ is the unique decomposition of P as a join of principal primes.

We sketch the existence of such a lattice L. Let F be a field, and let $R = F[\{X_\alpha \mid \alpha \in K\}]$ be the polynomial ring over a set of indeterminates indexed by by K. Since X satisfies ACC, it has maximal elements, say $\{M_\beta\}$. Since each M_β is a finite subset of K, we have a finitely generated prime ideal P_β generated by the X_α's with $\alpha \in M_\beta$. Let $S = R - \bigcup_\beta P_\beta$, so that S is a multiplicatively closed set in R. Let $T = R_S$; then, by Theorem 4.1 of [5], T is Noetherian. Consider the subset L of L(T) defined by all finite sums of products of ideals of the form $(X_\alpha)_S$. As in [2], it is easily seen that L is closed under joins, products, intersections, and residuals. Now the prime elements of L are precisely those ideals $(x_{\alpha 1}, \cdots, x_{\alpha n})_S$ where $(x_{\alpha 1}, \cdots, x_{\alpha n})_S \subseteq P_{\beta S}$ for some prime ideal P_β of R and the maximal elements of L are elements of the form $P_{\beta S} = (x_{\alpha 1}, \cdots, x_{\alpha n})_S$, where $\{\alpha_1, \cdots, \alpha_n\}$ is a maximal element M_β of X. Thus $L_{P_{\beta S}} \cong RL_n$, where $|M_\beta| = n$. Hence L is a regular distributive Noether lattice. A principal prime of L is of the form $(X_\alpha)_S$. Thus we have a bijection θ from the set of principal primes of L onto K. We can extend θ to an isomorphism $\hat{\theta}: \text{Spec}(L) \to X$ by defining $\hat{\theta}(P) = \{\theta(P_1), \cdots, \theta(P_n)\}$, where $P = P_1 \vee \cdots \vee P_n$ is the decomposition of P into a join of principal primes. Thus we have the following characterization of distributive regular Noether lattice domains.

THEOREM 8. *Let* L *be a distributive regular Noether lattice domain, and let* K *be the set of nonzero principal primes of* L. *The set* X = Spec(L) *may be considered as a subset of* FP(K), *the finite subsets of* K. *As a subset of* FP(K), X *satisfies the conditions*

(1) ACC,

(2) $Z \subseteq Y \in X$ *implies* $Z \in X$, *and*

(3) $\{P\} \in X$ *for every nonzero principal prime* P.

Conversely, suppose K is a set and $X \subseteq FP(K)$ satisfies the three conditions above. Then there exist a unique distributive regular Noether lattice domain L and a bijection θ from the set of nonzero principal primes of L onto K that extends to an isomorphism of posets $\hat{\theta}: \text{Spec}(L) \to X$ given by $\hat{\theta}(P) = \{\theta(P_1), \cdots, \theta(P_n)\}$, where $P = P_1 \vee \cdots \vee P_n$ is the unique decomposition of P into a join of nonzero principal primes.

THEOREM 9. *A distributive regular Noether lattice* L *is Noether-lattice embeddable into a regular Noetherian ring of the same Krull dimension.*

Proof. Since L is a finite direct product of domains, we may assume that L is a domain. Thus, by the previous construction, L is embeddable in the Noetherian domain T (with the notation as in the previous construction). It is clear that this embedding is actually a Noether lattice embedding. From the results of [5], it follows that the prime ideals $P_{\beta S}$ are precisely the maximal ideals of T. Hence T is locally regular and hence regular, and dim T = dim L.

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