NILPOTENT ELEMENTS OF COMMUTATIVE SEMIGROUP RINGS

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INTRODUCTION

All rings considered in this paper are assumed to be commutative, all semigroups are assumed to be abelian, the semigroup operation is written as addition, and the existence of a zero element with respect to this operation is assumed. On the other hand, the assumption that the rings under consideration have an identity plays no essential role, and therefore it will not be made.

We are concerned with the problem of determining the set of nilpotent elements of the semigroup ring of a semigroup S over a ring R. We follow the notation of D. G. Northcott [9, p. 128] and write R[X; S] for the semigroup ring of S over R; the elements of R[X; S] are "polynomials" $r_1 X^{s_1} + \cdots + r_n X^{s_n}$ in X with coefficients in R and exponents in S. If N is the nilradical of R, then it is clear that N[X; S] is contained in the nilradical of R[X; S]; this containment may be proper, and it depends upon the presence of certain torsion in the semigroup S.

After disposing of certain preliminaries concerning semigroups and semigroup rings in Section 1, we determine in Section 2 the nilradical of R[X; S] for the case where R is a ring of nonzero characteristic n. Let p be a prime integer; elements s and t of S are said to be *p-equivalent* if $p^k s = p^k t$ for some positive integer k, and S is *p-torsion-free* if distinct elements of S are not p-equivalent. In Theorem 2.5 we prove that the set of nilpotent elements of R[X; S] is N[X; S] + I, where I is the ideal generated by the set

$$\{rX^a - rX^b \mid r \in R, \text{for some prime divisor } p_i \text{ of } n \text{ the element } a \text{ is } p_i\text{-equivalent to } b, \text{ and a power of } p_i \text{ annihilates } r\};$$

thus N[X; S] is the nilradical of R[X; S] if and only if S is p_i-torsion-free for each prime divisor p_i of n.

In Section 3 we take up the case of a ring of characteristic 0. The existence of nilpotent elements of R[X; S] - N[X; S] is closely related to, but not equivalent to, the presence of asymptotically equivalent elements of S, where the definition is as follows. Elements a and b of S are *asymptotically equivalent* if there exists a positive integer n_0 such that na = nb for each $n \geq n_0$. If R is an integral domain, then R[X; S] has nonzero nilradical if and only if S contains nonidentical asymptotically equivalent elements. (Theorems 3.6 and 3.9.) In general, the nilradical of

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$R[X; S]$ will properly contain $N[X; S]$ if there exist distinct asymptotically equivalent elements s and t of S (assuming that R is not a nil ring), but the converse fails. In Theorem 3.14 and Propositions 3.15 and 3.16, we give a complete description of the nilradical of $R[X; S]$, for an arbitrary ring R and semigroup S.

1. PRELIMINARIES

If $(S, +)$ is a semigroup, then a *congruence* on S is an equivalence relation $\sim$ on S that is compatible with the semigroup operation on S — that is, $s \sim t$ implies that $s + u \sim t + u$ for each u. If $\sim$ is a congruence on S and if, for $x \in S$, $C[x]$ denotes the equivalence class determined by x, then the set of classes $C[x]$ forms a semigroup under the operation $C[x] + C[y] = C[x + y]$; this semigroup is denoted by $S/\sim$. The mapping $x \to C[x]$ is a semigroup homomorphism of S onto $S/\sim$. Conversely, each homomorphism ϕ of S onto a semigroup T determines a congruence $\sim$ on S defined by the rule that $s \sim t$ if and only if $\phi(s) = \phi(t)$. Moreover, T is isomorphic to the semigroup $S/\sim$ under the mapping $C[s] \leftrightarrow \phi(s)$. For a general reference on these matters, see [10].

If $\sim$ is a congruence on S, then we obtain a canonical homomorphism of the semigroup ring $R[X; S]$ onto $R[X; S/\sim]$; equivalently, if ϕ is a homomorphism of S onto a semigroup S_0, then the mapping $\sum_1^n r_i X^{s_i} \to \sum_1^n r_i X^{\phi(s_i)}$ of $R[X; S]$ onto $R[X; S_0]$ is the canonical homomorphism to which we refer. There is a dual to this result; namely, if μ is a homomorphism of the ring R onto a ring R_0, then the mapping $\sum_1^n r_i X^{s_i} \to \sum_1^n \mu(r_i) X^{s_i}$ is a homomorphism of $R[X; S]$ onto $R_0[X; S]$. In Sections 2 and 3, the kernels of these homomorphisms will be of interest. We record the result in Proposition 1.1.

(1.1) PROPOSITION. *Let* μ *be a homomorphism of* R *onto the ring* R_0, *with kernel* A, *and let* ϕ *be a homomorphism of* S *onto the semigroup* S_0. *Let the homomorphisms*

$$\mu^*: R[X; S] \to R_0[X; S], \qquad \phi^*: R[X; S] \to R[X; S_0], \qquad \tau: R[X; S] \to R_0[X; S_0]$$

be defined by the relations

$$\mu^*\left(\sum_{i=1}^n r_i X^{s_i}\right) = \sum_1^n \mu(r_i) X^{s_i}, \qquad \phi^*\left(\sum_1^n r_i X^{s_i}\right) = \sum_1^n r_i X^{\phi(s_i)},$$

$$\tau\left(\sum_1^n r_i X^{s_i}\right) = \sum_1^n \mu(r_i) X^{\phi(s_i)}.$$

Then

(1) *the kernel of* μ^* *is* $A[X; S]$,

(2) *the kernel of* ϕ^* *is the ideal* I *of* $R[X; S]$ *generated by*

$$\{rX^a - rX^b \mid r \in R \text{ and } \phi(a) = \phi(b)\},$$

and

(3) *the kernel of* τ *is* $A[X; S] + I$.

Proof. (1) is clear (and well known) and (3) follows from (1) and (2). In (2), it is clear that each $rX^a - rX^b$, and hence I, is contained in the kernel of ϕ^*. To prove the converse, we take a nonzero element $f = \sum_{i=1}^{m} f_i X^{s_i}$ in the kernel of ϕ^* and use induction on m. It is easy to see that $m = 1$ is impossible; the case $m = 2$ is obvious ($r_1 = -r_2$ and $\phi(s_1) = \phi(s_2)$). In the case $m > 2$, there are distinct exponents s_i and s_j of f such that $\phi(s_i) = \phi(s_j)$. The element $f - (f_i X^{s_i} - f_i X^{s_j})$ is in I, by the induction hypothesis, and hence f is also in I.

Two congruences on S will prove to be of importance in determining the nilradical of $R[X; S]$; here we introduce one of these, the p-congruence; we discuss the other, asymptotic equivalence, in Section 3.

(1.2) *Definition.* If p is a positive prime, then we define a relation $\underset{p}{\sim}$ on S by the rule that $s \underset{p}{\sim} t$ if there exists a positive integer k such that $p^k s = p^k t$. If $s \underset{p}{\sim} t$, then s and t are *p-equivalent;* if the relation $\underset{p}{\sim}$ on S is the trivial (identity) relation, then S is p-*torsion-free.*

The assertions of the next result are easily verified.

(1.3) PROPOSITION. *The relation* $\underset{p}{\sim}$ *is a congruence on* S, *and the semigroup* $S/\underset{p}{\sim}$ *is* p-*torsion-free. If* S *is* p-*torsion-free and if* $p^k s = p^k t$ *for some* $s, t \in S$ *and some positive integer* k, *then* $s = t$.

2. THE CASE OF A RING OF NONZERO CHARACTERISTIC

In determining the set of nilpotent elements of the semigroup ring $R[X; S]$, we consider first the case where the characteristic of R is nonzero. If $p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ is the prime factorization of the characteristic of R, then R is the direct sum of the ideals $R_1, R_2, \cdots, R_t$, where $R_i = \{x \in R \mid p_i^{e_i} x = 0\}$. Moreover,

$$R[X; S] = R_1[X; S] \oplus R_2[X; S] \oplus \cdots \oplus R_t[X; S],$$

and the nilradical of $R[X; S]$ is the direct sum of the nilradicals of the rings $R_i[X; S]$. Hence there is a sense in which we can reduce the problem to the consideration of rings of prime power characteristic.

(2.1) THEOREM. *Let the notation be as in the preceding paragraph, and let* N *be the nilradical of* R. *The following conditions are equivalent.*

(1) $N[X; S]$ *is the nilradical of* $R[X; S]$.

(2) *The semigroup* S *is* p_i-*torsion-free for each* i *such that* R_i *is not a nil ring.*

The proof of (2.1) uses a lemma and its corollary, each of which is a special case of (2.1) itself.

(2.2) LEMMA. *Assume that* R *is a ring of prime characteristic* p *and with nilradical* $N \neq R$. *Then* $N[X; S]$ *is the nilradical of* $R[X; S]$ *if and only if* S *is* p-*torsion-free.*

Proof. If S is p-torsion-free and if $f = \sum_{i=1}^{m} r_i X^{s_i}$ is nilpotent, then there exists a positive integer t such that $f^{p^t} = 0$. Therefore $0 = \sum_{i=1}^{m} r_i^{p^t} X^{p^t s_i}$, and since the exponents $p^t s_1, \cdots, p^t s_m$ are distinct, $r_i^{p^t} = 0$ for each i, and $f \in N[X; S]$, so that $N[X; S]$ is the nilradical of $R[X; S]$. Conversely, if S is not p-torsion-free, if s and t are distinct elements of S such that ps = pt, and if r is an element of R - N, then $rX^s - rX^t$ is a nilpotent element of $R[X; S]$ that is not in $N[X; S]$. This completes the proof of Lemma (2.2).

(2.3) COROLLARY. *If* D *is an integral domain of prime characteristic* p, *then the semigroup ring* $D[X; S]$ *has nilradical* (0) *if and only if* S *is* p-*torsion-free.*

Proof of Theorem 2.1. By virtue of the direct-sum decomposition $R = R_1 \oplus \cdots \oplus R_t$ and the induced direct decomposition of $R[X; S]$, it suffices to prove (2.1) in the case where R has prime-power characteristic p^k and $N \neq R$.

To prove that (2) implies (1), let f be a nilpotent element of $R[X; S]$, and let P be a proper prime ideal of R. The integral domain R/P has characteristic p, and hence $(R/P)[X; S]$ has nilradical (0), by Corollary 2.3. If ϕ is the canonical homomorphism of $R[X; S]$ onto $(R/P)[X; S]$, it follows that $\phi(f) = 0$; hence $f \in P[X; S]$, so that

$$f \in \bigcap_{\alpha} P_\alpha[X; S] = \left(\bigcap_{\alpha} P_\alpha\right)[X; S] = N[X; S],$$

where $\{P_\alpha\}$ is the set of proper prime ideals of R.

We prove, conversely, that (1) fails if S is not p-torsion-free. Thus assume that s and t are distinct elements of S such that ps = pt, and choose an element r in R - N. Then $(rX^s - rX^t)^p$ belongs to the ideal $pR[X; S]$ of $R[X; S]$. Since $R[X; S]$ has characteristic p^k, it follows that $(rX^s - rX^t)^{pk} = 0$. Therefore $rX^s - rX^t$ is a nilpotent element of $R[X; S]$ that is not in $N[X; S]$.

Theorem 2.1 enables us to determine the nilradical of $R[X; S]$ in the case of a ring of nonzero characteristic. This characterization is contained in Theorem 2.5, which is an immediate consequence of the next result.

(2.4) THEOREM. *Let* R *be a ring of prime-power characteristic* p^n. *The nilradical of* $R[X; S]$ *is the ideal* $N[X; S] + I$, *where* N *is the nilradical of* R *and* I *is the ideal of* $R[X; S]$ *generated by* $\{rX^a - rX^b \mid r \in R$ *and* $a \underset{p}{\sim} b\}$.

Proof. By passage to the ring $R[X; S]/N[X; S] \simeq (R/N)[X; S]$, we can assume that N = (0). Consider the homomorphism of semigroup rings

$$\phi: R[X; S] \to R[X; S/\underset{p}{\sim}].$$

The kernel I of ϕ is precisely the set of nilpotent elements of $R[X; S]$. For if f is nilpotent, then, since $S/\underset{p}{\sim}$ is p-torsion-free, Theorem 2.1 implies that $\phi(f) = 0$, hence $f \in I$. Conversely, the proof of Theorem 2.1, together with Proposition 1.1, shows that each element of I is nilpotent. Hence $N[X; S] + I$ is the nilradical of $R[X; S]$, as asserted.

(2.5) THEOREM. *Assume that the ring* R *has nonzero characteristic* n, *and let the ideals* $R_1, \cdots, R_t$ *be defined as in the first paragraph of this section. The nilradical of the semigroup ring* $R[X; S]$ *is*

$$N[X; S] + \sum_{i=1}^{t} R_i\{rX^a - rX^b \mid r \in R_i \text{ and } a \underset{p_i}{\sim} b\},$$

where N *is the nilradical of* R.

3. NILPOTENT ELEMENTS IN SEMIGROUP RINGS OF CHARACTERISTIC 0

We turn to the problem of determining the set of nilpotent elements of $R[X; S]$ in the case where R is a ring of characteristic 0. Our first result uses the concept of a *Hilbert ring,* defined to be a ring with identity in which each proper prime ideal is an intersection of maximal ideals [4], [6, Section 1.3], [3, Section 31] (the terminology of [8] and [1, Section 3.4] is *Jacobson ring*).

(3.1) *Definition.* A ring T with identity is an FMR-*ring* if for each maximal ideal M of T, the residue-class field T/M is finite.

(3.2) PROPOSITION. *Let* T *be an* FMR-*ring. The following conditions are equivalent.*

(1) T *is a Hilbert ring.*

(2) *Each finitely generated ring extension of* T *is an* FMR-*ring.*

Proof. Assume that T is a Hilbert ring, and let X be an indeterminate over T. The polynomial ring $T[X]$ is a Hilbert ring [4], [8], and we prove that $T[X]$ is also an FMR-ring. Thus, if M is a maximal ideal of $T[X]$, then $M_0 = M \cap T$ is a maximal ideal of T [4]. Since M contains $M_0[X]$, we can reduce the problem to the case where T is a finite field, and in that case it is clear that $T[X]$ is an FMR-ring. If $T[\alpha]$ is any simple ring extension of T, then $T[\alpha]$ is a homomorphic image of $T[X]$, and hence $T[\alpha]$ is a Hilbert FMR-ring. By induction, it follows that each finitely generated ring extension of T is a Hilbert FMR-ring. At any rate, (1) implies (2).

Conversely, if T is not a Hilbert ring, then there exists a maximal ideal M of $T[X]$ such that $M \cap T = P$ is a nonmaximal prime ideal of T. Thus T/P is a domain, not a field, and hence is not finite. Since T/P is isomorphic to a subring of $T[X]/M$, it follows that $T[X]$ is not an FMR-ring. This completes the proof of Proposition 3.2.

(3.3) THEOREM. *Let* R *be a ring of characteristic* 0, *and assume that* S *is torsion-free. Then* $N[X; S]$ *is the nilradical of* $R[X; S]$, *where* N *is the nilradical of* R.

Proof. Let R^* be the ring obtained by canonically adjoining an identity element to R [3, p. 5]. Since R has characteristic 0, the ideal N is also the nilradical of R^*. Thus it suffices to prove that the nilradical of $R^*[X; S]$ is $N[X; S]$. Let $\sum_{i=1}^{n} r_i X^{s_i}$ be a nilpotent element of $R^*[X; S]$. Proposition 3.2 implies that the ring $R_0 = Z[r_1, \cdots, r_n]$, where Z is the subring of R^* generated by the identity element of R^*, is a Hilbert FMR-ring. It follows that the nilradical N_0 of R_0 is $\bigcap M_\lambda$, where $\{M_\lambda\}$ is the set of maximal ideals of R_0. Moreover, $f \in R_0[X; S]$, and Corollary 2.3 implies that $f \in M_\lambda[X; S]$ for each λ, since the characteristic of R_0/M_λ is nonzero. Consequently, $f \in \bigcap_\lambda M_\lambda[X; S] = N_0[X; S] \subseteq N[X; S]$, as we wished to prove.

If S is not torsion-free, then $R[X; S]$ may have nonzero nilpotent elements, even if R is an integral domain of characteristic 0. This is due to the possible existence of distinct elements a and b of S such that $na = nb$ for almost all positive integers n. Our next results are directed toward a determination of the effect on the nilradical of $R[X; S]$ of the existence of such elements a and b. If the relation $\sim$ is defined on S by the rule that $a \sim b$ if and only if there exists a positive integer n_0 such that $na = nb$ for each $n \geq n_0$, then $\sim$ is a congruence on S; we call elements a and b of S such that $a \sim b$ *asymptotically equivalent.* If distinct elements of S are not asymptotically equivalent, then we say that S is *free of asymptotic torsion.* We proceed to show that $S/\sim$ is free of asymptotic torsion.

(3.4) LEMMA. *Let* a *and* b *be positive integers with greatest common divisor* 1. *If* $n \geq (a - 1)(b - 1)$, *then there exist nonnegative integers* x *and* y *such that* $n = xa + yb$.

Proof. Without loss of generality, assume that $0 < a < b$. Write $ib = q_i a + r_i$, where $0 \leq r_i < a$ and $0 \leq i \leq a - 1$. Since b is a unit modulo a, the set $\{r_i\}_{i=0}^{a-1}$ is a complete set of residues modulo a. If $n \geq (a - 1)(b - 1)$, write $n = ta + r$, where $0 \leq r < a$, and hence $r \in \{r_i\}_{i=0}^{a-1}$. If $r = r_j$, then $t \geq q_j$; for $t < q_j$ implies that $ta + r_j < q_j a + r_j = jb$, which contradicts the facts that $jb - n$ is divisible by a and $jb - (a - 1)(b - 1) \leq a - 1$. Write $t = q_j + c$, where c is nonnegative. Then $n = ca + q_j a + r_j = ca + jb$.

We remark that alternate forms of Lemma 3.4 appear in [7, Theorem 1.4.1], [10, Theorem 82], and [5].

(3.5) PROPOSITION. *If* $\sim$ *is the relation of asymptotic equivalence on the semigroup* S, *then the semigroup* $S/\sim$ *is free of asymptotic torsion.*

Proof. Assume that $C[a]$ and $C[b]$ are asymptotically equivalent elements of $S/\sim$, and let n_0 be such that $nC[a] = nC[b]$ for each $n \geq n_0$—that is, $na \sim nb$ for each $n \geq n_0$. We must show that if $na \sim nb$ for each $n \geq n_0$, then $a \sim b$. If $n_1 \geq n_0$, then $n_1 a \sim n_1 b$. Therefore there exists k_1 such that $k_1 n_1 a = k_1 n_1 b$. Choose $n_2 \geq n_0$ such that n_2 is relatively prime to $k_1 n_1$. Then there exists a positive integer k_2, relatively prime to $k_1 n_1$, such that $k_2 n_2 a = k_2 n_2 b$. It follows that $n_1 k_1$ and $n_2 k_2$ are relatively prime, that $n_1 k_1 a = n_1 k_1 b$, and that $n_2 k_2 a = n_2 k_2 b$. By Lemma 3.4, there exists an integer c_0 such that each $c \geq c_0$ can be written as $t(n_1 k_1) + m(n_2 k_2)$, where t and m are nonnegative. Therefore

$$ca = [t(n_1 k_1) + m(n_2 k_2)]a = t(n_1 k_1 a) + m(n_2 k_2 a)$$

$$= t(n_1 k_1 b) + m(n_2 k_2 b) = [t(n_1 k_1) + m(n_2 k_2)]b = cb.$$

It follows that $a \sim b$ and $C[a] = C[b]$.

(3.6) THEOREM. *Let* R *be a ring. The ideal of* $R[X; S]$ *generated by the set*

$$\{rX^a - rX^b \mid r \in R \text{ and } a \text{ is asymptotically equivalent to } b\}$$

consists of nilpotent elements.

Proof. It suffices to show that $rX^a - rX^b$ is nilpotent if a is asymptotically equivalent to b. There exists n_0 such that $na = nb$ for each $n \geq n_0$. If $m = 2n_0 + 1$ and if u and v are nonnegative integers such that $u + v = m$, then either $u > n_0$ or $v > n_0$, and hence $ua = ub$ or $va = vb$. In either case, $ua + vb = (u + v)a = ma$. Canonically adjoin an identity e to R. Then

$$(rX^a - rX^b)^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (rX^a)^{m-i} (rX^b)^i$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} r^m X^{(m-i)a+ib} = eX^{ma}\left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} r^m \right)$$

$$= eX^{ma}(r - r)^m = 0.$$

(3.7) LEMMA. *Assume that* D *is an integral domain of characteristic* 0 *and that* $\{P_\lambda\}_{\lambda\in\Lambda}$ *is a family of prime ideals of* D *such that* $\bigcap_\lambda P_\lambda = (0)$. *Let* $\{p_i\}_{i=1}^n$ *be a finite set of prime integers, and let* $\{P_{\lambda_0}\}_{\lambda_0\in\Lambda_0}$ *be the subset of* $\{P_\lambda\}_{\lambda\in\Lambda}$ *such that the characteristic of* D/P_{λ_0} *is distinct from each* p_i. *Then* $\bigcap_{\lambda_0\in\Lambda_0} P_{\lambda_0} = (0)$.

Proof. Let $a \in \bigcap_{\lambda_0\in\Lambda_0} P_{\lambda_0}$. Then $p_1 p_2 \cdots p_n a = 0$, since $p_i D \subseteq P_\lambda$ if the characteristic of D/P_λ is p_i. Therefore $a = 0$, for otherwise the characteristic of D would divide $p_1 p_2 \cdots p_n$.

For a result closely related to Lemma 3.7, see [2, Lemma 4].

(3.8) LEMMA. *Let* D *be a Hilbert* FMR-*domain of characteristic* 0. *If* S *is free of asymptotic torsion, then* 0 *is the only nilpotent element of* D[X; S].

Proof. We first observe that if s and t are distinct elements of S, then $s \underset{p}{\sim} t$ for at most one prime integer p. For if $s \underset{p}{\sim} t$ and $s \underset{q}{\sim} t$, where p and q are distinct primes, then there exist positive integers u and v such that $p^u s = p^u t$ and $q^v s = q^v t$. Since p^u and q^v are relatively prime, Lemma 3.4 guarantees that s and t are asymptotically equivalent, a contradiction.

Assume that $f = \sum_{i=1}^n f_i X^{s_i}$ is nilpotent. There are at most finitely many primes p such that $s_i \underset{p}{\sim} s_j$ for some distinct exponents s_i and s_j of f. Let this set of primes be $\{p_i\}_{i=1}^m$. Since D is a Hilbert ring, $\bigcap_{\lambda\in\Lambda} M_\lambda = (0)$, where $\{M_\lambda\}_{\lambda\in\Lambda}$ is the family of maximal ideals of D. By Lemma 3.7,

$$\bigcap \{M_{\lambda_0} \mid \text{the characteristic of } D/M_{\lambda_0} \text{ is not in the set } \{p_i\}_{i=1}^m\} = (0).$$

Suppose that the characteristic $q > 0$ of D/M_{λ_0} is not in the set $\{p_i\}_{i=1}^m$, and let f^* denote the image of f under the canonical homomorphism of D[X; S] onto $(D/M_{\lambda_0})[X; S]$. Since f is nilpotent, f^* is nilpotent — say $(f^*)^{q^k} = 0$. But

$$(f^*)^{q^k} = \left(\sum_{i=1}^n f^* X^{s_i} \right)^{q^k} = \sum_{i=1}^n (f^*)^{q^k} X^{q^k s_i},$$

and $q^k s_i \neq q^k s_j$ for $i \neq j$ since q is not in the set $\{p_i\}_{i=1}^m$. It follows that $f \in M_{\lambda_0}[X; S]$. Therefore $f \in \bigcap\{M_{\lambda_0}[X; S]\} = \left(\bigcap M_{\lambda_0}\right)[X; S] = (0)$, and this completes the proof of (3.8).

(3.9) THEOREM. *Assume that* D *is an integral domain of characteristic* 0. *If* S *is free of asymptotic torsion, then* 0 *is the only nilpotent element of* D[X; S].

Proof. Let K be the quotient field of D. It suffices to prove that K[X; S] has no nonzero nilpotent elements. Suppose that $f = \sum_{i=1}^n f_i X^{s_i}$ is nilpotent. Let $D_0 = Z[f_1, f_2, \cdots, f_n]$, where Z is the integral domain generated by the identity element of K. Then D_0 is a Hilbert FMR-domain of characteristic 0 and f is a nilpotent element of $D_0[X; S]$. By Lemma 3.8, f = 0.

(3.10) COROLLARY. *If* S *is cancellative and if* D *is a domain of characteristic* 0, *then* D[X; S] *has no nonzero nilpotent elements.*

Proof. If S is cancellative, then S is free of asymptotic torsion.

(3.11) COROLLARY. *Assume that* D *is a domain of characteristic* 0. *The nilradical of* D[X; S] *is generated by the set*

$$\{dX^a - dX^b \mid d \in D \textit{ and } a \textit{ is asymptotically equivalent to } b\}.$$

Proof. In view of Theorem 3.6, we need only show that each nilpotent element f of D[X; S] is in the ideal $(\{dX^a - dX^b\})$. Let ~ be the relation of asymptotic equivalence on S, let ϕ be the canonical homomorphism of S onto S/~, and let ϕ^* be the induced homomorphism of D[X; S] onto D[X; S/~]:

$$\phi^*\left(\sum_{i=1}^m d_i X^{s_i}\right) = \sum_{i=1}^m d_i X^{\phi(s_i)}.$$

By Proposition 3.5, distinct elements of S/~ are not asymptotically equivalent, and hence by Theorem 3.9, D[X; S/~] has nilradical (0). Therefore f is in the kernel of ϕ^*, and Proposition 1.1 shows that $\{dX^a - dX^b \mid d \in D \text{ and } a \sim b\}$ generates the kernel of ϕ^*.

We have determined the nilradical of D[X; S] for the case where D is a domain of characteristic 0. Our next results are directed toward the case where R is an arbitrary ring of characteristic 0.

(3.12) LEMMA. *Let* $f = \sum_{i=1}^m f_i X^{s_i}$ *be an element of* R[X; S] *such that no two of the distinct exponents of* f *are* p-*equivalent for the fixed prime* p. *Assume that* $f \in P[X; S] + I$, *where* P *is a prime ideal of* R *with* R/P *of characteristic* p, *and where* I *is the ideal of* R[X; S] *generated by the set*

$$\{rX^a - rX^b \mid r \in R \textit{ and } a \underset{p}{\sim} b\}.$$

Then $f \in P[X; S]$.

Proof. Let ϕ^* be the canonical homomorphism of R[X; S] onto $R[X; S/\underset{p}{\sim}]$, where $\underset{p}{\sim}$ is the p-congruence on S. Then $\phi^*(f) \in P[X; S/\sim]$, since the kernel of ϕ^* is generated by the set $\{rX^a - rX^b \mid r \in R \text{ and } a \underset{p}{\sim} b\}$. Moreover,

$\phi^*(f) = \sum_{i=1}^m f_i X^{\phi(s_i)}$, and since the exponents $\phi(s_1), \cdots, \phi(s_m)$ are distinct by assumption, each f_i is in P. Therefore f is in $P[X; S]$.

(3.13) LEMMA. *Let* $f = \sum_{i=1}^m f_i X^{s_i} \in R[X; S]$, *and let* P *be a prime ideal of* R *such that* R/P *has characteristic* $p > 0$. *If the positive integer* k *is such that* $s_i \underset{p}{\sim} s_j$ *implies that* $p^k s_i = p^k s_j$, *then the relation* $f^{p^k} \in P[X; S] + I$ *implies that* $f^{p^k} \in P[X; S]$, *where* I *is defined as in Lemma* 3.12.

Proof. Let μ be the canonical homomorphism of $R[X; S]$ onto $(R/P)[X; S]$; the element $\mu(f^{p^k}) = [\mu(f)]^{p^k}$ belongs to the ideal

$$\mu(I) = \{r^* X^a - r^* X^b \mid r^* \in R/P \text{ and } a \underset{p}{\sim} b\}$$

of $(R/P)[X; S]$, and in $(R/P)[X; S]$, it satisfies the hypothesis of Lemma 3.12 (where the ideal P of that result is replaced by the zero ideal of R/P). Therefore $\mu(f^{p^k}) = 0$ and $f^{p^k} \in P[X; S]$, as asserted.

(3.14) THEOREM. *Let* R *be a commutative ring, and let* $\{P_\lambda\}_{\lambda \in \Lambda}$ *be the family of prime ideals of* R. *Then the nilradical of* $R[X; S]$ *is*

$$\bigcap_{\lambda \in \Lambda} \{P_\lambda[X; S] + I_{p_\lambda}\},$$

where p_λ *is the characteristic of* R/P_λ, *and where* I_{p_λ} *is the ideal of* $R[X; S]$ *generated by the set* $\{rX^a - rX^b \mid r \in R$ *and* $a \underset{p_\lambda}{\sim} b\}$; *if* $p_\lambda = 0$, *then* I_{p_λ} *is the ideal of* $R[X; S]$ *generated by the set* $\{rX^a - rX^b \mid r \in R$ *and* $a \sim b\}$.

Proof. Let N be the nilradical of R. Since $N[X; S] + I_0$ is contained in $P_\lambda[X; S] + I_{p_\lambda}$ for each λ, we can pass to the ring $(R/N)[X; S/\sim]$. Thus, without loss of generality we can assume that the nilradical of R is zero and that asymptotically equivalent elements of S are identical.

Assume that f is a nilpotent element of $R[X; S]$. The ring

$$R[X; S]/(P_\lambda[X; S] + I_{p_\lambda}) \simeq (R/P_\lambda)[X; S/\underset{p_\lambda}{\sim}]$$

has nilradical (0); if $p_\lambda > 0$, this follows from Corollary 2.3, and if $p_\lambda = 0$, from Theorem 3.9. Therefore $f \in \bigcap_{\lambda \in \Lambda} \{P_\lambda[X; S] + I_{p_\lambda}\}$.

Conversely, assume that the element $f = \sum_{i=1}^m r_i X^{s_i}$ belongs to $\bigcap_{\lambda \in \Lambda} \{P_\lambda[X; S] + I_{p_\lambda}\}$. If s and t are distinct exponents in the canonical form of f, then s and t are equivalent for at most one prime integer p; for if $s \underset{p}{\sim} t$ and $s \underset{q}{\sim} t$, where p and q are distinct primes, then by Lemma 3.4, s and t are asymptotically equivalent and hence $s = t$. It follows that there is a finite set $\{p_i\}_{i=1}^n$ of primes such that if s_i and s_j are q-equivalent for some $i \neq j$, then q is in

$\{p_i\}_{i=1}^n$. Choose a positive integer k large enough so that if $s_i \underset{p}{\sim} s_j$, where $i \neq j$, then $p^k s_i = p^k s_j$. We prove that $f^t = 0$, where $t = (p_1 p_2 \cdots p_n)^k$. If R/P_λ has characteristic $q \notin \{p_i\}_{i=1}^n$, then $f \in P_\lambda[X; S]$, by Lemma 3.12. If R/P_λ has characteristic $q \in \{p_i\}_{i=1}^n$, then $f^t = (f^{q^k})^{t_1}$, where $q = p_i$ and $t_1 = (p_1 \cdots p_{i-1} p_{i+1} \cdots p_n)^k$. Denote by $g \to g^*$ the canonical homomorphism of $R[X; S]$ onto $(R/P_\lambda)[X; S]$. Then in $(R/P_\lambda)[X; S]$,

$$(f^{q^k})^* = (f^*)^{q^k} = \left(\sum_{i=1}^m r_i^* X^{s_i}\right)^{q^k} = \sum_{i=1}^m (r_i^*)^{q^k} X^{q^k s_i}.$$

By choice of k, no two of the distinct exponents of $(f^*)^{q^k}$ are q-equivalent. Clearly, $(f^*)^{q^k}$ belongs to the nilradical of $(R/P_\lambda)[X; S]$, which is

$$(\{rX^a - rX^b \mid r \in R/P_\lambda \text{ and } a \underset{q}{\sim} b\}),$$

by Theorem 2.4; therefore

$$(f^*)^{q^k} \in (0^*) + (\{rX^a - rX^b \mid r \in R/P_\lambda \text{ and } a \underset{q}{\sim} b\}).$$

Since no two of the exponents of $(f^*)^{q^k}$ are q-equivalent, $(f^*)^{q^k} = 0^*$, by Lemma 3.12. Therefore $f^{q^k} \in P_\lambda[X; S]$, and in any case, $f^t \in P_\lambda[X; S]$ for each λ. Since $\bigcap_{\lambda \in \Lambda} P_\lambda[X; S] = (0)$, it follows that $f^t = 0$, and this completes the proof of Theorem 3.14.

Theorem 3.14 characterizes the nilpotent elements of the semigroup ring $R[X; S]$ for a commutative ring R and an abelian semigroup S. Our final results provide alternate descriptions of the ideal $\bigcap_{\lambda \in \Lambda} \{P_\lambda[X; S] + I_{p_\lambda}\}$ that are, in some cases, easier to apply.

(3.15) PROPOSITION. *Let the hypothesis and notation be as in Theorem* 3.14. *Then*

$$\bigcap_{\lambda \in \Lambda} \{P_\lambda[X; S] + I_{p_\lambda}\} = \bigcap_{i=0}^{\infty} \{A_{p_i}[X; S] + I_{p_\lambda}\},$$

where $p_0 = 0$, p_n *is the* n*th prime integer,*

$$A_{p_i} = \bigcap_{\lambda \in \Lambda} \{P_\lambda \mid R/P_\lambda \textit{ has characteristic } p_i\},$$

I_0 *is the ideal of* $R[X; S]$ *generated by the set*

$$\{rX^a - rX^b \mid r \in R \textit{ and } a \textit{ is asymptotically equivalent to } b\},$$

and I_{p_i} *is the ideal of* $R[X; S]$ *generated by the set* $\{rX^a - rX^b \mid r \in R$ *and* $a \underset{p_i}{\sim} b\}$.

Proof. Write $\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i$, where $\Lambda_i = \{\lambda \in \Lambda \mid R/P_\lambda$ has characteristic $p_i\}$. Then

$$\bigcap_{\lambda \in \Lambda} \{P_\lambda[X; S] + I_{p_\lambda}\} = \bigcap_{i=0}^{\infty} \left(\bigcap_{\lambda \in \Lambda_i} \{P_\lambda[X; S] + I_{p_i}\} \right),$$

so that we need only establish the equality $\bigcap_{\lambda \in \Lambda_i} \{P_\lambda[X; S] + I_{p_i}\} = A_{p_i}[X; S] + I_{p_i}$. Since I_{p_i} is the kernel of the canonical homomorphism of $R[X; S]$ onto $R[X; S/\underset{p_i}{\sim}]$, this amounts to establishing the equality $\bigcap_{\lambda \in \Lambda_i} \{P_\lambda[X; S/\underset{p_i}{\sim}]\} = A_{p_i}[X; S/\underset{p_i}{\sim}]$; but this equality follows immediately from the fact that $A_{p_i} = \bigcap_{\lambda \in \Lambda_i} P_\lambda$.

(3.16) PROPOSITION. *Let the hypothesis and notation be as in Theorem* 3.14. *The nilradical of* $R[X; S]$ *is*

$$\sum_{\substack{\pi \subseteq \mathscr{P} \\ |\pi| < \infty}} \left\{ (C_\pi[X; S] + I_0) \cap \left[\bigcap_{p \in \pi} (A_p[X; S] + I_p) \right] \right\},$$

where $\mathscr{P}$ *is the set of positive prime integers, where*

$$C_\pi = \bigcap_{\lambda \in \Lambda} \{P_\lambda \mid \textit{the characteristic of } R/P_\lambda \textit{ is distinct from each prime in } \pi\},$$

and where I_0, A_p, *and* I_p *are defined as in Proposition* 3.15.

Proof. In view of Theorem 3.14 and Proposition 3.15, it is clear that the set $\sum_{\substack{\pi \subseteq \mathscr{P} \\ |\pi| < \infty}} \cdots$ is contained in the nilradical of $R[X; S]$.

Conversely, if f is nilpotent, then $f \in \bigcap_{\lambda \in \Lambda} \{P_\lambda[X; S] + I_{p_\lambda}\}$, and the proof of Theorem 3.14 shows that f belongs to

$$(C_\pi[X; S] + I_0) \cap \left\{ \bigcap_{p \in \pi} (A_p[X; S] + I_p) \right\},$$

where $q \in \pi$ if and only if there exist distinct exponents s and t in the canonical form of f such that $s \underset{q}{\sim} t$.

Proposition 3.16 indicates how examples may be constructed to show that Theorem 3.9 does not generalize to rings with zero divisors. Thus, let R be the weak direct sum of the family $\{GF(p_i)\}_{i=1}^{\infty}$ of prime fields, where $p_1 < p_2 < p_3 < \cdots$ is the sequence of primes, and let $S = \{0, g\}$ be the cyclic group of order 2; R is a ring of characteristic 0, and S is free of asymptotic torsion; but the nilradical of $R[X; S]$ can easily be shown to be $\{0, e_1 X^0 - e_1 X^g\}$, where e_1 is the element $(1, 0, 0, \cdots)$ of R. In the notation of Proposition 3.16, $e_1 X^0 - e_1 X^g$ is in $C_\pi[X; S] \cap I_{p_1}$, where $\pi = \{p_1\} = \{2\}$.

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