A PROBLEM IN THE CONFORMAL GEOMETRY OF CONVEX SURFACES

Dimitri Koutroufiotis

1. INTRODUCTION

A homothety of E^3 in the sense of elementary Euclidean geometry is a mapping of the form $\vec{x} \to \lambda \vec{x}$; here $\vec{x}$ and λ denote a position vector and a constant. For subsets A and $\overline{A}$ of E^3 we call a mapping $\Phi: A \to \overline{A}$ a *homothety* if it is the restriction to A of a homothety of E^3.

Let S and $\overline{S}$ denote two smooth (C^∞), oriented surfaces in E^3. Suppose that there exists a diffeomorphism Φ between them such that at points and directions corresponding to each other under Φ, the normal curvatures k and $\overline{k}$ satisfy an equation $\overline{k} = ck$, where c is a constant depending on Φ, but neither on position nor on direction. If, in addition, S is not a developable surface and has nowhere-dense umbilics (points where the principal curvatures coincide), then Φ is a homothety up to a Euclidean motion. This local result, which actually holds, *mutatis mutandis*, for hypersurfaces in a space of constant curvature, is a trivial generalization of a theorem due to R. S. Kulkarni [5, p. 95]. It would be of interest to investigate whether a similar statement can be made in case the constant c is replaced by a smooth function ϕ satisfying appropriate assumptions. In this paper we shall show that if S and $\overline{S}$ are ovaloids (that is, compact surfaces in E^3 with positive Gaussian curvature), then the condition $\overline{k} = \phi k$ does indeed imply that S and $\overline{S}$ are essentially homothetic, provided we impose on ϕ a certain mild restriction. Several local and global questions arise naturally; we shall discuss some of them at the end.

We introduce some additional terminology. Let S and $\overline{S}$ be smooth, two-dimensional Riemannian (or pseudo-Riemannian) manifolds. A diffeomorphism $\Phi: S \to \overline{S}$ will be called *conformal* if there exists a smooth function $\phi \neq 0$ on S, the *scale function,* with the property $\langle \Phi_* \alpha, \Phi_* \beta \rangle_{\Phi(P)} = \phi(P) \langle \alpha, \beta \rangle_P$ for all points P in S and all vectors α and β in the tangent space S_P. If (u, v) is a pair of local parameters for S, we may carry it over to $\overline{S}$, using Φ, so that corresponding points are described by the same pair (u, v). We may then say, equivalently, that Φ is conformal if the quadratic forms Λ and $\overline{\Lambda}$ corresponding to the metrics on S and $\overline{S}$ satisfy the condition $\overline{\Lambda} = \phi\Lambda$ in these parameters. In the case of surfaces in E^3, "conformal" with no further specification will always mean conformal with respect to their first fundamental forms.

Received August 12, 1974.
Partially supported by NSF Grant GP-42833.

Michigan Math. J. 22 (1975).

2. THE MAIN RESULT

We shall first prove three auxiliary propositions of independent interest.

The following lemma may be viewed as a generalization of the well-known proposition that the only convex surfaces with conformal spherical-image mapping are the spheres. See [2] for this and related results.

LEMMA 1. *Let* S *and* $\bar{S}$ *be connected and orientable surfaces in* E^3 *with positive Gaussian curvature. Assume that there exists a conformal diffeomorphism* Φ *between them with the property that at corresponding points the normals are parallel. Then* Φ *is a homothety, modulo a reflection and a translation.*

Proof. If a certain quantity on S is denoted by a certain symbol, the same symbol with a bar denotes the same quantity on S. Consider a point P on S that either is not an umbilic or is an interior point of umbilics (in other words, if P is umbilic, let S be spherical near P). We may introduce line-of-curvature parameters (u, v) near P, and the three fundamental forms of S near P then read [3, p. 63]

$$I = E\,du^2 + G\,dv^2, \qquad II = k_1 E\,du^2 + k_2 G\,dv^2, \qquad III = k_1^2 E\,du^2 + k_2^2 G\,dv^2,$$

where the orientation on S is chosen so that $k_1 \geq k_2 > 0$ for the principal curvatures.

We use the diffeomorphism in question to parametrize also $\bar{S}$ by (u, v) so that corresponding points are characterized by equal parameter values. By assumption, $\bar{I} = \phi^2 I$ for a certain positive and smooth function ϕ on S. Let

$$\overline{II} = \bar{L}\,du^2 + 2\bar{M}\,du\,dv + \bar{N}\,dv^2.$$

After a possible reflection on $\bar{S}$, we may assume that also $\overline{II}$ is positive definite. Since $\overline{III} = (\bar{k}_1 + \bar{k}_2)\overline{II} - \bar{k}_1\bar{k}_2\bar{I}$ and $\overline{III} = III$ by assumption, we obtain the equation $(\bar{k}_1 + \bar{k}_2)\bar{M} = 0$, hence $\bar{M} = 0$. This means that (u, v) is a line-of-curvature parameter pair also on $\bar{S}$, hence $\bar{L} = \phi^2 E \bar{k}_1$ and $\bar{N} = \phi^2 G \bar{k}_2$. From the equations $\overline{III} = III$ we now deduce that

$$k_1 = \phi\bar{k}_1, \qquad k_2 = \phi\bar{k}_2. \tag{2.1}$$

It follows that $\overline{II} = \phi II$ near P, and by continuity this must be true on the whole of S. The Codazzi equations in these parameters are (see [3, p. 63])

$$\left.\begin{aligned} \frac{\partial k_1}{\partial v} &= (k_2 - k_1)\frac{E_v}{2E} \\ \frac{\partial k_2}{\partial u} &= (k_1 - k_2)\frac{G_u}{2G} \end{aligned}\right\} \quad \text{on } S, \tag{2.2}$$

$$\left.\begin{aligned} \frac{\partial \bar{k}_1}{\partial v} &= (\bar{k}_2 - \bar{k}_1)\frac{(\phi^2 E)_v}{2\phi^2 E} \\ \frac{\partial \bar{k}_2}{\partial u} &= (\bar{k}_1 - k_2)\frac{(\phi^2 G)_u}{2\phi^2 G} \end{aligned}\right\} \quad \text{on } \bar{S}. \tag{2.3}$$

Substituting in (2.3) for $\bar{k}_1$ and $\bar{k}_2$ their expressions from (2.1) and using (2.2), we obtain the equations $\phi_u k_1 = 0$ and $\phi_v k_2 = 0$, hence ϕ is constant near P. Therefore, ϕ is constant in a neighbourhood of every point on S, with the possible exception of a nowhere-dense set of umbilics. Now, if P_1 and P_2 are two points on S and C is a smooth curve connecting them, then by considering ϕ on C we see by a standard argument that $\phi(P_1) = \phi(P_2)$, so that ϕ is constant on S.

If S is described in E^3 by the vector function $\vec{x}$, let S^* be the surface, homothetic to S, described by $\phi\vec{x}$; then $I^* = \phi^2 I = \bar{I}$ and $II^* = \phi II = \overline{II}$. By the Uniqueness Theorem of surface theory (which holds also in the large for connected surfaces (see [3, p. 119]), we conclude that S^* can be made to coincide with $\bar{S}$ by a proper Euclidean motion, which, in fact, must be a translation since S^* and $\bar{S}$ have parallel normals at corresponding points.

Note that Lemma 1 is not true if we drop the restriction that the curvatures of S and $\bar{S}$ be positive; witness pairs of minimal surfaces with the same spherical image.

LEMMA 2. *Let* Ψ *be a conformal diffeomorphism of the unit sphere* Σ *in* E^3 *with scale function* F. *If* Ψ *is not an isometry, then* F *has exactly two critical points; they are both nondegenerate and occur at antipodes.*

Proof. Composing Ψ, if need be, with a reflection, we may assume that Ψ is properly conformal. Without loss of generality, we may further assume that Ψ leaves the north pole P_1 fixed and that F is critical at P_1. Let P_2 be the south pole. We shall show that P_1 is the only critical point of F on $\Sigma - \{P_2\}$ and that it is nondegenerate. This will imply immediately that P_2 is the only other critical point of Ψ. Nondegeneracy of P_2 is now proved in exactly the same manner as for P_1. By means of sterographic projection of Σ from P_1 onto the equatorial plane (x, y), we represent Ψ as the transformation

$$z \to a + \lambda z \qquad (\lambda \neq 0,\ z = x + iy)$$

of the extended complex plane onto itself, since $\infty \to \infty$ by construction. We may assume λ to be a positive number, since multiplication of $|\lambda|$ with a unimodular complex number amounts to a rotation of Σ about the north-south axis.

If we parametrize the sphere by means of stereographic projection from the south pole onto the equatorial plane, the same mapping Ψ is now represented by the transformation

$$z \to f(z) = \frac{z}{\bar{a}z + \lambda}.$$

We can easily verify this by observing that if a point on Σ is mapped on z under stereographic projection from P_1, then that same point is mapped on $1/\bar{z}$ under stereographic projection from P_2. The point P_1 now has coordinate $z = 0$. In terms of these coordinates, the standard metric of Σ is given by the equation

$$(2.4) \qquad ds^2 = \frac{4}{(1 + |z|^2)^2} |dz|^2 \qquad (dz = dx + i\,dy).$$

This formula can easily be verified by means of the parametric representation of Σ in [1, p. 18], for example.

Under Ψ, the line element (2.4) at the point z is transformed into the line element

$$ds^2 = \frac{4}{(1+|f(z)|^2)^2}\left|\frac{df}{dz}\right|^2 |dz|^2 = \frac{4}{\left(1+\left|\frac{z}{\bar{a}z+\lambda}\right|^2\right)^2}\left|\frac{\lambda}{(\bar{a}z+\lambda)^2}\right|^2 (dx^2+dy^2),$$

at the point with coordinate f(z). Now F^{-1} is given by the ratio of (2.4) to (2.5):

$$\left(\frac{|\bar{a}z+\lambda|^2+|z|^2}{1+|z|^2}\right)^2 \frac{1}{\lambda^2}. \tag{2.6}$$

At z = 0, the function (2.6) has a critical point, by assumption. Computing the partial derivatives at z = 0 and setting them equal to zero, we see that $\Re(\lambda a) = \Im(\lambda a) = 0$, hence a = 0. Now (2.6) reduces to the function

$$\psi(z) = \frac{1}{\lambda^2}\left(\frac{\lambda^2+|z|^2}{1+|z|^2}\right)^2.$$

Setting $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0$, we see that $(1-\lambda^2)z = 0$, hence z = 0 since λ cannot be 1. Furthermore, $H(\psi) = \psi_{xx}\psi_{yy} - \psi_{xy}^2 > 0$ at z = 0. This completes the proof of Lemma 2.

COROLLARY. *Let* Ψ *be a conformal diffeomorphism of* Σ *with scale function* F. *If at some critical point of* F *the value of* F *is* 1, *then* Ψ *is an orthogonal transformation.*

Proof. Note first that an isometry of Σ is the restriction to Σ of an orthogonal transformation of E^3; this fact is a special case of the Congruence Theorem for ovaloids [3, p. 129]. Now, if Ψ were not an isometry, F would have exactly two critical points, by Lemma 2: its absolute maximum and its absolute minimum. Thus, the point P where F = 1 and dF = 0 would be an extremum, say the maximum, and $1 - F > 0$ except at P. Consider now the metric ds^2 given by (2.4) and the metric $F\,ds^2$; they both have Gaussian curvature 1. Therefore, by the Gauss-Bonnet theorem [3, p. 47],

$$4\pi = \int_\Sigma d\omega = \int_\Sigma F\,d\omega,$$

where $d\omega$ stands for the area element with respect to ds^2. It follows that $1 - F$ must change sign on Σ, which is a contradiction.

Lemma 2 can also be deduced from a general result by W. O. Vogel on concircular mappings of Riemannian manifolds [9, p. 237, Korollar 4].

The following lemma generalizes Kulkarni's Lemma 1 in [5]:

LEMMA 3. *Let* S *and* $\bar{S}$ *be oriented surfaces in* E^3, *and let* $S \xrightarrow{\Phi} \bar{S}$ *be a diffeomorphism. Denote by* $k(P, \alpha)$ *the normal curvature of* S *at* P *in the direction* α. *Let* $\phi \neq 0$ *be a smooth real-valued function on* S. *If* S *has nowhere-dense umbilics and* $k(\Phi(P), \Phi_*(\alpha)) = \phi(P)\cdot k(P, \alpha)$ *for all* P *and* α, *then* Φ *is* I-*conformal and* II-*conformal.*

Proof. Take a point $P \in S$ that is not an umbilic. Introduce line-of-curvature parameters u and v on S near P, and carry them over to $\overline{S}$ via Φ: the assumptions imply that u and v are again line-of-curvature parameters on $\overline{S}$, since principal directions correspond under Φ. Denote by k_1 and k_2 ($k_1 \neq k_2$) the principal curvatures of S, and by $\overline{k}_1$ and $\overline{k}_2$ those of $\overline{S}$ in the corresponding directions. Then $\overline{k}_1 = \phi k_1$, $\overline{k}_2 = \phi k_2$, and

$$I = E\,du^2 + G\,dv^2, \qquad \overline{I} = \overline{E}\,du^2 + \overline{G}\,dv^2,$$

$$II = k_1 E\,du^2 + k_2 G\,dv^2, \qquad \overline{II} = \phi(k_1 \overline{E}\,du^2 + k_2 \overline{G}\,dv^2).$$

Recall that the normal curvature k of a surface S at the point P and in the direction $\alpha = (du, dv)$ is defined by

$$k(P, \alpha) = \frac{II(P;\, du,\, dv)}{I(P;\, du,\, dv)}.$$

We may assume without loss of generality that $E = G = 1$ at P. By hypothesis and construction, we have at P the equation

$$\frac{\phi(k_1 \overline{E}\,du^2 + k_2 \overline{G}\,dv^2)}{\overline{E}\,du^2 + \overline{G}\,dv^2} = \phi\,\frac{k_1\,du^2 + k_2\,dv^2}{du^2 + dv^2}$$

for any direction $du : dv$. Cross-multiplying and simplifying, we see that

$$(k_1 - k_2)(\overline{G} - \overline{E})\,du^2\,dv^2 = 0,$$

hence $\overline{G} = \overline{E}$ at P. Thus, $\overline{I}/I$ is independent of direction at P, and therefore $\overline{I} = \phi_1 I$ for some smooth positive function ϕ_1 and $\overline{II} = \phi_2 II$, with $\phi_2 = \phi\phi_1$ on an everywhere-dense set of points on S. By continuity, this is true on the whole of S.

Note that, conversely, $\overline{I} = \phi_1 I$ and $\overline{II} = \phi_2 II$ imply $\overline{k} = \phi_2 \phi_1^{-1} k$.

We are now ready to prove the main result.

THEOREM 1. *Let* S *and* $\overline{S}$ *be oriented ovaloids, let* S *have nowhere-dense umbilics, let* $S \xrightarrow{\Phi} \overline{S}$ *be a diffeomorphism, and let* ϕ *be a smooth point-function on* S *with the properties:*

(a) $k(\Phi(P), \Phi_*(\alpha)) = \phi(P) \cdot k(P, \alpha)$ *for all points* P *and directions* α;

(b) *not all local extrema of* ϕ *occur at umbilics.*

Then ϕ *is constant and* Φ *is a homothety, modulo a Euclidean motion.*

Proof. We may assume that S and $\overline{S}$ are oriented by interior normals. It follows that $\phi > 0$. By Lemma 3, we see that $\overline{I} = \phi_1 I$ and $\overline{II} = (\phi\phi_1) II$ for some smooth $\phi_1 > 0$. Furthermore, $\overline{H} = \phi H$ (mean curvatures) and $\overline{K} = \phi^2 K$ (Gaussian curvatures); therefore

$$\overline{III} = 2(\phi H)(\phi\phi_1\, II) - (\phi^2 K)(\phi_1\, I) = (\phi^2 \phi_1)\, III,$$

and the induced mapping $\Psi \mid \Sigma$ between the spherical images of S and $\overline{S}$ — which by Hadamard's theorem [3, p. 54] is a diffeomorphism -- is also conformal. Consider first the case where $\Psi \mid \Sigma$ is an isometry, that is, the restriction to Σ of an orthogonal transformation Ψ on E^3. The mapping $\Psi^{-1} \circ \Phi: S \to \Psi^{-1}(\overline{S})$ is clearly a I-conformal diffeomorphism between S and $\Psi^{-1}(\overline{S})$ with the property that at corresponding points the normals are parallel. Therefore Lemma 1 applies, and Φ is essentially a homothety. We shall complete the proof by showing that the assumption that Ψ is not an isometry contradicts hypothesis (b) of the theorem.

We claim first that the functions ϕ_1, $\phi\phi_1$, and $\phi^2\phi_1$ have the same critical points. To see this, introduce isothermic parameters (x, y) locally on S, so that $I = E(dx^2 + dy^2)$, and write the Codazzi equations for S and $\overline{S}$ in these parameters [3, p. 79]. Making use of $\overline{I} = \phi_1 I$ and $\overline{II} = \phi_2 II$, where $\phi_2 = \phi\phi_1$, we obtain after some straightforward manipulations the system

$$(2.7)\qquad \begin{aligned} L \frac{\partial}{\partial y}(\log \phi_2) - M \frac{\partial}{\partial x}(\log \phi_2) &= EH \frac{\partial}{\partial y}(\log \phi_1), \\ M \frac{\partial}{\partial y}(\log \phi_2) - N \frac{\partial}{\partial x}(\log \phi_2) &= -EH \frac{\partial}{\partial x}(\log \phi_1), \end{aligned}$$

whence we deduce that ϕ_1 and ϕ_2 have the same critical points, since $K \neq 0$. Clearly, if ϕ_1 and ϕ_2 are critical at P, then so is $\phi_2^2/\phi_1 = \phi^2\phi_1$. Conversely, if ϕ_2^2/ϕ_1 is critical at P, then (2.7) shows that grad $\phi_1 = 0$ at P; we shall not use this in the proof, however.

Assume now that $\Psi \mid \Sigma$ is not an isometry, and consider a nonumbilic point P where ϕ has a local extremum. If we introduce line-of-curvature parameters near P and use $\overline{k}_i = \phi k_i$ (i = 1, 2), we easily obtain from the Codazzi equations (2.2) for S and $\overline{S}$ near P

$$(2.8)\qquad \begin{aligned} \frac{\partial}{\partial u}(\log \phi) &= \frac{1}{2}\left(\frac{k_1}{k_2} - 1\right)\frac{\partial}{\partial u}(\log \phi_1), \\ \frac{\partial}{\partial v}(\log \phi) &= \frac{1}{2}\left(\frac{k_2}{k_1} - 1\right)\frac{\partial}{\partial v}(\log \phi_1). \end{aligned}$$

Thus, ϕ_1 is also critical at P, and since $\phi^2\phi_1$ has only two critical points, by Lemma 2, so does ϕ_1. Therefore ϕ_1 has an extremum at P, and for its Hessian determinant we have the inequality $H(\phi_1) \geq 0$ at P. Recall that $H(\phi) \geq 0$ at P, by our choice of P. Now, if we eliminate ϕ from the equations (2.8), we obtain the equation

$$\frac{\partial^2 \phi_1}{\partial u\, \partial v} = 0 \qquad \text{at } P;$$

therefore, again by virtue of equations (2.8),

$$\frac{\partial^2 \phi}{\partial u\, \partial v} = 0 \qquad \text{at } P.$$

If we differentiate the first equation of (2.8) with respect to u and the second with respect to v and multiply the ensuing equations, we now obtain at P the relation

$$H(\phi) = -c^2 H(\phi_1) \qquad (c \neq 0).$$

Therefore $H(\phi) = H(\phi_1) = 0$ at P; hence $H(\phi^2\phi_1) = 0$ at P, which contradicts Lemma 2, according to which $H(\phi^2\phi_1) > 0$ at both critical points.

3. REMARKS AND QUESTIONS

(i) Perhaps the condition (b) of Theorem 1 is not necessary for the conclusion to hold. Counterexamples are lacking. In any case, the proof implies that if there exist pairs of nonhomothetic ovaloids S and $\bar{S}$ admitting a diffeomorphism with $\bar{I} = \phi_1 I$ and $\bar{II} = \phi_2 II$, then ϕ_1 and ϕ_2 must have the same critical points, exactly two in number, which must be antipodes (that is, points with parallel normals) on S, mapped onto antipodes on $\bar{S}$. In addition, each type of extremum is attained by ϕ_1 and ϕ_2 at the same point: this last assertion follows from the formula

$$\left(\frac{\phi_2^2}{\phi_1} - 1\right) K = -\frac{1}{2}\Delta(\log \phi_1), \tag{3.1}$$

(where Δ denotes the Laplace-Beltrami operator with respect to I), since the maximum of $\phi_2^2\phi_1^{-1}$ is greater than 1 and occurs at a critical point of both ϕ_1 and ϕ_2. One deduces (3.1) readily from the Theorema Egregium for S and $\bar{S}$.

The proof of Theorem 1 does not go through, if one relaxes the assumption of strict convexity ($K > 0$) to $K \geq 0$. Whether it remains valid more generally for arbitrary compact sufraces is, of course, also not known to the author.

(ii) Does Theorem 1 remain true if we replace the word "ovaloids" in it by "pieces of strictly convex surfaces"? We may formulate this problem as follows: does there exist a strictly convex surface S with

$$I = E(du^2 + dv^2) \quad \text{and} \quad II = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

and with a nonumbilic point P on it such that the system of partial differential equations

$$Lg_v - Mg_u = \left(\frac{L+N}{2}\right) f_u, \qquad Mg_v - Ng_u = -\left(\frac{L+N}{2}\right) f_v, \qquad \Delta f = 2EK(1 - e^{2g-f})$$

has a nonconstant solution (f, g) in the vicinity of P? Here Δ signifies the ordinary Laplacian in the (u, v)-plane. One verifies easily that, should such a solution exist, the quadratic forms $\bar{I} = e^f I$ and $\bar{II} = e^g II$ define a surface $\bar{S}$ in E^3, hence the mapping $S \to \bar{S}$ by equal parameters satisfies the condition $\bar{k} = \phi k$ but is not a homothety, since ϕ is not constant.

(iii) Quite generally, mappings between S and $\bar{S}$ satisfying both conditions $\bar{I} = \phi_1 I$ and $\bar{II} = \phi_2 II$ seem to have been investigated for the first time by P. Stäckel, who called them *conformal-conjunctive* in [6, p. 560]. There exist surfaces admitting local, conformal-conjunctive automorphisms that are not homotheties. One such class consists of all surfaces with $K < 0$ whose asymptotic lines intersect at a constant angle θ [7, pp. 490-497]. Actually, Stäckel later showed [8, pp. 10-12] that up to stretchings and motions, there exists exactly one such surface for each θ ($0 < \theta < \pi/2$), namely the surface of revolution

$$
(3.2)\qquad
\begin{aligned}
x &= (\cosh u)^{\mu} \cos v\,, \\
y &= (\cosh u)^{\mu} \sin v\,, \\
z &= \sqrt{\mu} \int \sqrt{\mu - (\mu - 1)(\cosh u)^2}\,(\cosh u)^{\mu - 1}\, du\,,
\end{aligned}
$$

where $\mu = \cot^2(\theta/2)$. In the limiting case $\theta = \pi/2$ (minimal surfaces), equations (3.2) are those of the catenoid. Of course, in this case we can easily see directly, using line-of-curvature parameters, that each small piece of a minimal surface without flat points can be mapped conformal-conjunctively onto any other such surface.

It is also easy to construct examples of pairs of nonhomothetic, complete, developable surfaces ($K \equiv 0$), with no flat points, and admitting a global, conformal-conjunctive diffeomorphism that is not a homothety [5, p. 100].

In the case of convex surfaces, however, the only examples known to the author of pairs admitting nontrivial conformal-conjunctive mappings are furnished by pairs of spheres. Are there any others?

(iv) We can view two ovaloids S and $\bar{S}$ as Riemann surfaces of genus 1 in the usual way, namely by restricting our attention to certain isothermal parameter systems on them. The Uniformization Theorem, therefore, guarantees the existence of a I-conformal diffeomorphism between them. If we orient S and $\bar{S}$ by interior normals, their second fundamental forms define new Riemann-surface structures on them (biisothermal parameters; see [4], for example), hence, again by the Uniformization Theorem, there exist diffeomorphisms between them that are II-conformal. We may ask whether there exist pairs of ovaloids admitting diffeomorphisms that are *both* I-conformal and II-conformal. Note that every pair of homothetic surfaces admits such a mapping with $\phi_1 = \text{const}$, $\phi_2 = \text{const}$, and $\phi_2^2\phi_1^{-1} = 1$. We shall now state a converse of this fact, which may be interpreted as saying that if a diffeomorphism is I-conformal and II-conformal and a homothety up to second order at a single point, then it is a homothety everywhere.

THEOREM 2. *Let* S *and* $\bar{S}$ *be ovaloids, and let* $S \xrightarrow{\Phi} \bar{S}$ *be a diffeomorphism with* $\bar{I} = \phi_1 I$ *and* $\overline{II} = \phi_2 II$. *Assume in addition that there exists a point where* $\phi_2^2\phi_1^{-1} = 1$ *and* $d(\phi_2^2\phi_1^{-1}) = 0$. *Then* Φ *is a homothety.*

Proof. As in the proof of Theorem 1, the induced mapping Ψ between the spherical images of S and $\bar{S}$ is conformal with scale function $\phi_2^2\phi_1^{-1}$. Therefore, by the corollary to Lemma 2, Ψ is an isometry and Φ is a homothety.

Quite likely, some other normalization instead of the point-normalization in this theorem (some integral-normalization, say) will again ensure that Φ is a homothety. It is clear from the proof that this amounts to finding normalizations on a conformal mapping Ψ of the unit sphere that will guarantee that Ψ is in fact an orthogonal transformation. It is conceivable, however, that no normalization is necessary if we assume in addition that the nonumbilics on S are dense.

REFERENCES

1. L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable.* Second edition, McGraw-Hill, New York-Toronto, 1966.

2. N. Hicks, *Connexion preserving, conformal, and parallel maps.* Michigan Math. J. 10 (1963), 295-302.

3. H. Hopf, *Lectures on differential geometry in the large.* Mimeographed notes by J. W. Gray, Stanford University, 1956.

4. T. Klotz, *Some uses of the second conformal structure on strictly convex surfaces.* Proc. Amer. Math. Soc. 14 (1963), 793-799.

5. R. S. Kulkarni, *Congruence of hypersurfaces.* J. Differential Geometry 8 (1973), 95-102.

6. P. Stäckel, *Ueber Abbildungen.* Math. Ann. 44 (1894), 553-564.

7. ———, *Beiträge zur Flächentheorie.* Leipziger Berichte, Math.-Phys. Classe, 48 (1896), 478-504.

8. ———, *Beiträge zur Flächentheorie.* Leipziger Berichte, Math.-Phys. Classe, 50 (1898), 3-20.

9. W. O. Vogel, *Einige Kennzeichnungen der homothetischen Abbildungen eines Riemannschen Raumes unter den kreistreuen Abbildungen.* Manuscripta Math. 9 (1973), 211-228.

University of California, Santa Barbara
Santa Barbara, California 93016