UNIFORM ALGEBRAS CONTAINING THE REAL AND IMAGINARY PARTS OF THE IDENTITY FUNCTION

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A uniform algebra on $\Gamma = \{z: |z| = 1\}$ is a subalgebra of $C(\Gamma)$ that is closed under the topology of the supremum norm, contains the constants, and separates the points of Γ. The canonical example is the disk algebra A, which is the uniform algebra consisting of all functions in $C(\Gamma)$ that extend continuously to $\{z: |z| \leq 1\}$ to be analytic on $D = \{z: |z| < 1\}$. In a recent paper [4], J. M. F. O'Connell shows that if B is a uniform algebra with $\Re B = \Re A$, then there exists a homeomorphism Φ of Γ onto Γ such that

$$B = A \circ \Phi = \{f \circ \Phi : f \in A\}.$$

W. P. Novinger [3] generalizes this result to the setting in which it is only assumed that $\Re B \supseteq \Re A$. He shows that in this case either $B = C(\Gamma)$ or $B = A \circ \Phi$ for some homeomorphism Φ. We show that to obtain the latter conclusion, it is sufficient to assume that $\Re B$ contains the real and imaginary parts of the identity function Z.

THEOREM 1. *Let* B *be a uniform algebra on* Γ *such that* $\Re B$ *contains* $\Re Z$ *and* $\Im Z$. *Then either* $B = C(\Gamma)$ *or there exists a homeomorphism* Φ *of* Γ *onto* Γ *such that* $B = A \circ \Phi$.

Proof. By hypothesis, there exist functions ψ and ϕ in B such that $\Re\psi = \Re Z$ and $\Im\phi = \Im Z$.

Case 1. Either ψ or ϕ is one-to-one on Γ. We shall assume that ψ is one-to-one on Γ. The proof for the case where ϕ is one-to-one is similar. Let W denote the interior of the Jordan curve $\psi(\Gamma)$. Let f denote the Riemann mapping of W onto D; then f extends continuously to $\overline{W}$, mapping $\psi(\Gamma)$ homeomorphically onto Γ. By Mergelyan's theorem, f can be uniformly approximated by polynomials on $\psi(\Gamma)$, and thus $\Phi = f \circ \psi$ is in B. Hence, $A \circ \Phi \subseteq B$, or equivalently, $A \subseteq B \circ \Phi^{-1}$. Applying Wermer's maximality theorem to the uniform algebra $B \circ \Phi^{-1}$, we see that either $B \circ \Phi^{-1} = C(\Gamma)$ or $B \circ \Phi^{-1} = A$. It follows immediately that $B = C(\Gamma)$ or $B = A \circ \Phi$.

Before proceeding to Case 2, we shall establish some useful results.

LEMMA 1. *Let* B *be a uniform algebra on* Γ *containing functions* ψ *and* ϕ *with* $\Re\psi = \Re Z$ *and* $\Im\phi = \Im Z$. *If* $\psi(z_1) = \psi(z_2)$ *or* $\phi(z_1) = \phi(z_2)$ *and* E_1 *and* E_2 *are the two closed subarcs of* Γ *with end points* z_1 *and* z_2, *then* E_1 *and* E_2 *are peak sets for* B. *Furthermore,* $B | E_j$ *is a closed subalgebra of* $C(E_j)$ *for* $j = 1, 2$.

Proof. If $z_1 = z_2$, then the conclusion is trivial. We shall assume that $\psi(z_1) = \psi(z_2)$. If $\phi(z_1) = \phi(z_2)$, the proof is similar. Note that our assumption implies that $z_2 = \bar{z}_1$.

Let K be the union of $\psi(\Gamma)$ and the bounded components of $\mathbb{C} - \psi(\Gamma)$. There exists a closed rectangle R containing $\psi(E_2)$ such that one edge of R is contained in $\{z: \Re z = \Re\psi(z_1)\}$. Let f be the Riemann mapping of int R onto D; then f

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extends to a homeomorphism of R onto $\overline{D}$. We can assume that $f(\psi(z_1)) = 1$. For $w \in K - R$, let $f(w) = 1$. Since f is continuous on K and analytic on int K, we can approximate f uniformly by polynomials, on K, by Mergelyan's theorem. Hence, $f \circ \psi$ is in B and it peaks on E_1. A similar argument shows that E_2 is a peak set for B. By [1, p. 163], $B \mid E_j$ is a closed subalgebra of $C(E_j)$ for $j = 1, 2$.

LEMMA 2. *Let* B *be a uniform algebra on* Γ, *let* X *be a compact subset of* Γ, *and let* f *be in* B. *If* f(X) *is contained in a Jordan arc and* $\Re f$ *is one-to-one on* X, *then* $B \mid X$ *is dense in* C(X).

Proof. By Mergelyan's theorem, the function $w \to \Re w$ can be uniformly approximated by polynomials on f(X), and we conclude that $\Re f \mid X$ is in $B \mid X$. By the generalized Stone-Weierstrass theorem, $\Re f \mid X$ and the constants generate a dense subalgebra of C(X). Hence, $B \mid X$ is dense in C(X).

The following is a theorem of R. E. Mullins [2, p. 272].

THEOREM 2. *Let* B *be a uniform algebra on a compact metric space* X. *Let* $F_1, \cdots, F_n$ *be* n *closed sets such that*

$$X = \bigcup_{i=1}^{n} F_i \quad \text{and} \quad B \mid F_i = C(F_i) \quad (i = 1, \cdots, n).$$

Then B = C(X).

In the following, our goal is to show that Γ is the union of sets to which we can apply Lemmas 1 and 2 and Theorem 2, and to conclude that $B = C(\Gamma)$.

Case 2 *of Theorem* 1. Neither ψ nor ϕ is one-to-one on Γ.

Proof. Consider first the situation where $\psi(i) = \psi(-i)$. Let E_1 and E_2 be the two closed subarcs of Γ with end points i and -i. The image $\phi(E_j)$ is a Jordan arc, for $j = 1, 2$. Hence, Lemma 2 implies that $B \mid E_j$ is dense in $C(E_j)$, for $j = 1, 2$. Applying Lemma 1, we see that $B \mid E_j$ is closed in $C(E_j)$ for $j = 1, 2$. Thus, $B \mid E_j = C(E_j)$ for $j = 1, 2$. Now we can invoke Theorem 2 to obtain the conclusion $B = C(\Gamma)$. A similar proof shows that $B = C(\Gamma)$ if $\phi(1) = \phi(-1)$.

Hence, we can assume that there exist z_1 and z_2 satisfying the four conditions.

(1) $\Re z_1 < 0 < \Re z_2$,

(2) $\Im z_j \geq 0$ for $j = 1, 2$,

(3) $\psi(z_j) = \psi(\bar{z}_j)$ for $j = 1, 2$, and

(4) $\psi(\bar{z}) \neq \psi(z)$ for each z between z_1 and z_2 on the upper half-circle.

Let E_1 and E_2 be the two closed subarcs of Γ with end points z_1 and $\bar{z}_1$, and let E_3 and E_4 be the two closed subarcs of Γ with end points z_2 and $\bar{z}_2$. We make a similar construction for ϕ, noting particularly that in this case $\Im w_1 < 0 < \Im w_2$ and $\phi(w_j) = \phi(-\bar{w}_j)$ for $j = 1, 2$. Label the second collection of subarcs F_1, F_2, F_3, and F_4. We can assume that E_1, E_3, F_1, and F_3 all have length less than π. Applying Lemma 1 to ψ and E_1 and Lemma 2 to ϕ and E_1, we see that $B \mid E_1 = C(E_1)$. Similarly, we deduce that $B \mid E_3 = C(E_3)$, $B \mid F_3 = C(F_3)$, and $B \mid F_1 = C(F_1)$.

Let $K = E_2 \cap E_4 \cap F_2 \cap F_4$ and note that $\Gamma = E_1 \cup E_3 \cup F_1 \cup F_3 \cup K$. If K is empty, we apply Theorem 2 and deduce that $B = C(\Gamma)$. If K is not empty, then K is a peak set for B, since it is the intersection of four peak sets. Hence, $B \mid K$ is closed [1, p. 163]. Because $\psi(K)$ is properly contained in the Jordan curve

$\psi(E_2 \cap E_4)$, it follows by Mergelyan's theorem that $\Re\psi \mid K$ is in $B \mid K$. Similarly, $\Im\phi \mid K$ is in $B \mid K$. Applying the generalized Stone-Weierstrass theorem to the closed algebra generated by $\Re\psi \mid K$, $\Im\phi \mid K$, and the constants, we see that $B \mid K = C(K)$. Hence, Theorem 2 implies that $B = C(\Gamma)$.

Remarks. Novinger's result follows easily after Theorem 2. Note that for each compact set X properly contained in Γ, there exists a conformal automorphism of D whose real part is one-to-one on some arc containing X, and thus $B \mid X$ is dense in $C(X)$ by Lemma 2. If ψ is not one-to-one on Γ, then apply Lemma 1 to ψ and invoke Theorem 2 to obtain the conclusion $B = C(\Gamma)$.

In the proof of Theorem 1 we actually prove more than we state. In Case 1, we prove that if there exists ψ in B such that $\Re\psi = \Re Z$ and ψ is one-to-one, then the conclusion of Theorem 1 holds. In Case 2, we show that if either ψ or ϕ is not one-to-one, then $B = C(\Gamma)$.

COROLLARY. *Let* B *be a uniform algebra on* Γ, *and let* τ *be the boundary-value function of a conformal automorphism of* D *onto* D. *If* $\Re B$ *contains* $\Re\tau$ *and* $\Im\tau$, *then either* $B = C(\Gamma)$ *or there exists a homeomorphism* Φ *of* Γ *onto* Γ *such that* $B = A \circ \Phi$.

Proof. Note that $\Re B \circ \tau^{-1}$ contains $\Re Z$ and $\Im Z$, and then apply Theorem 1 to $B \circ \tau^{-1}$.

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