EXISTENCE OF SOLUTIONS OF A NONLINEAR PROBLEM IN POTENTIAL THEORY

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1. INTRODUCTION

In this paper we study the nonlinear boundary value problem

$$(1)\qquad \begin{cases} \Delta u + g(x, y, u) = 0, & (x, y) \in A = \{x^2 + y^2 < 1\}, \\ u = 0, & (x, y) \in \partial A = \{x^2 + y^2 = 1\} \end{cases}$$

under various hypotheses on g. We denote by N the Nemitsky operator defined by $Nu = -g(x, y, u(x, y))$, and we assume that $N: S \to S$ maps the space $S = L_2(A)$ into itself. We obtain the following results:

I. If $N: S \to S$ is monotone and continuous, then the nonlinear problem (1) has a unique solution.

II. If $-N: S \to S$ is monotone, continuous, and bounded, and if the Gateaux derivative of N "lies between two consecutive eigenvalues" of the associated linear problem

$$\Delta u + \lambda u = 0 \text{ in } A, \qquad u = 0 \text{ on } \partial A,$$

then the problem (1) has a unique solution.

III. If $Nu = \lambda_m u + h(x, y, u)$, where λ_m is an eigenvalue of the associated linear problem (the resonance case), and the Nemitsky operator $M: S \to S$ defined by $Mu = h(x, y, u)$ is continuous and bounded in S, then under suitable hypotheses the nonlinear problem (1) has at least one solution.

This paper was motivated by a paper of L. Cesari [3] concerning problem (1), where use is made of the alternative method by means of which the problem is reduced to an equivalent system of two operator equations. This method, which has its origin in Lyapunov and Schmidt's work, was formulated by Cesari [2] in functional-analytic terms. The method was then applied by Cesari and several other authors to a wide variety of situations (see J. K. Hale [6]). In this paper we follow this method, but we appeal to several concepts of nonlinear functional analysis, namely maximal monotone operators, nonlinear Hammerstein equations, and Schauder's principle of invariance of domain.

The chief feature of this method is that one can handle problems of the type (1) where the linear operator has a nontrivial nullspace (which is the case if for example $g(x, y, u) = \lambda u + h(x, y, u)$, where λ is an eigenvalue of the associated linear problem $\Delta u + \lambda u = 0$ in A and $u = 0$ on ∂A). As will be obvious from the proofs, the nonlinear problem (1) could be stated in a more general form for elliptic problems in more general domains. However, for the sake of simplicity, we shall restrict ourselves to problem (1).

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In [3], Cesari proved that problem (1) has a solution if g is locally Lipschitzian and satisfies suitable growth hypotheses. Problems of type (1) have been studied for the existence of solutions under various hypotheses on g by A. Hammerstein [7] and C. L. Dolph [4]. In particular, Dolph obtains results similar to II. The case when N is monotone was also studied by H. Brézis, M. G. Crandall, and A. Pazy [1]. We give different proofs, so that it becomes easy to see the natural extension of our method to the case of perturbation at resonance (which was not studied by the authors mentioned above). The resonance case, in particular situations, has recently been the subject of study by several authors. The first results in this direction, for elliptic problems, was obtained by E. M. Landesman and A. C. Lazer [11] (this was a generalization of a corresponding result for ordinary differential equations by A. C. Lazer and D. E. Leach [12]). More recently, M. Nakao [15], M. Schatzman [16], and S. Fučik [5] have also studied the resonance situation. In this paper we illustrate, by giving separate proofs for the nonresonance situations, how the alternative method adapts itself to the resonance situation.

2. THE PROJECTION METHOD OF CESARI

Let S be the Hilbert space $L_2(A)$, that is, the space of all functions $u(x, y)$ that are measurable and L^2-integrable in A. Let $\langle\ ,\ \rangle$ and $\|\cdot\|$ denote the usual inner product and the norm in S, respectively. The linear problem

$$\text{(2)} \qquad \begin{aligned} &\Delta u + \lambda u = 0 \qquad ((x, y) \in A), \\ &u = 0 \qquad ((x, y) \in \partial A) \end{aligned}$$

has fundamental systems $\{\lambda_i\}$ and $\{\phi_i\}$ of eigenvalues and orthonormal eigenfunctions with $0 < \lambda_1 < \lambda_2 < \cdots$. Also, $\{\phi_i\}$ is a complete orthonormal system in $L_2(A)$. Thus every element $u \in S$ has a Fourier series

$$u(x, y) = \sum c_i \phi_i \qquad (c_i = \langle u, \phi_i \rangle).$$

Let $S_0 = \{\phi_1, \cdots, \phi_m\}$ $(m \geq 1)$, and let $P: S \to S_0$ be the projection operator defined by

$$Pu = \sum_1^m c_i \phi_i .$$

Let S' be the subset of S of all functions $u(x, y)$ that are essentially bounded in A. Also, let X be the set of all functions $u(x, y)$ such that $u(x, y)$ is continuous in $A \cup \partial A$ with $u = 0$ on ∂A, u has continuous first-order partial derivatives in A, and u (computed in the sense of distributions) is a measurable, essentially bounded function defined a. e. in A. Then it can be seen that

$$X \subset S', \qquad \Delta: X \to S' \subset S, \qquad \text{and } H: S' \to X,$$

where H is the linear operator defined by

$$Hu = -\sum_1^\infty c_i \lambda_i^{-1} \phi_i \qquad \text{for each } u = \sum_1^\infty c_i \phi_i \in S.$$

We now have the relations

$$(3)\qquad \begin{cases} \Delta Hu = u & \text{for all } u \in S', \\ H(I - P)\Delta u = (I - P)u, & \\ \Delta Pu = P\Delta u & \text{for all } u \in X. \end{cases}$$

Now, if $u = \sum c_i \phi_i$, then

$$H(I - P)u = -\sum_{m+1}^{\infty} c_i \lambda_i^{-1} \phi_i .$$

Also, for $u(x, y) \in S$, we have the relation

$$(4)\quad \langle -H(I - P)u, u \rangle = \sum_{m+1}^{\infty} c_i{}^2 \lambda_i^{-1} \geq \lambda_{m+1} \sum_{m+1}^{\infty} c_1{}^2 \lambda_i^{-2} = \lambda_{m+1} \|-H(I - P)u\|^2 .$$

Let N be the Nemitsky operator defined by

$$Nu = -g(x, y, u(x, y))$$

for all (x, y) in A and all u in S. Also, let $N: S \to S'$. If $u(x, y) \in S$ is a solution of the nonlinear problem (1), then, by applying the operator $H(I - P)$ to both sides of the equation $\Delta u = Nu$ and using (3), we see that $(I - P)u = H(I - P)Nu$, in other words, that

$$(5)\qquad u - H(I - P)Nu = Pu .$$

Now, if $u \in X$ is a solution of (5), then by applying Δ to both sides of (5), we obtain the equation

$$\Delta u - (I - P)Nu = \Delta Pu = P\Delta u .$$

Thus $P(\Delta u - Nu) = \Delta u - Nu$. Hence a solution $u \in X$ of (5) is a solution of (1) if and only if

$$(6)\qquad P(\Delta u - Nu) = 0 .$$

Thus we have reduced the problem of solving (1) to that of solving the system of equations (5) and (6), respectively. If however the equation

$$(7)\qquad u - H(I - P)Nu = x^*$$

has a unique solution u for each $x^* \in S_0$, then $Pu = x^*$ and the operator $[I - H(I - P)N]^{-1}$ is a single-valued operator over S_0. Equation (6) could then be rewritten in the form

$$PN[I - H(I - P)N]^{-1}x^* - P\Delta u = 0 .$$

By virtue of (3), this reduces to

(8) $$PN[I - H(I - P)N]^{-1}x^* - \Delta x^* = 0.$$

We can now state the following result: If there exists an $x^* \in S_0$ for which (7) is uniquely solvable and (8) is solvable, then the corresponding solution u of (7) is a solution of (1).

Hence, in order to obtain existence results for problem (1), it is sufficient to consider the system of equations (7) and (8) respectively. Equations (7) and (8) are called the *auxiliary* and *bifurcation* equations, respectively.

3. EXISTENCE THEOREMS

We first consider the case when the nonlinear operator N is monotone. As we mentioned in the introduction, even though the results of this case are known, we give a proof, which is later naturally extended to the case of perturbations at resonance. We recall the following definitions.

An operator $N: S \to S$ is said to be *monotone* if $\langle Nu - Nv, u - v \rangle \geq 0$ for all $u, v \in S$.

An operator $N: S \to S$ is said to be *strictly monotone* if there exists $c > 0$ such that $\langle Nu - Nv, u - v \rangle \geq c \|u - v\|^2$ for all $u, v \in S$.

An operator $N: S \to S$ is said to be *maximal monotone* if it is monotone and is maximal in the family of monotone mappings from S into 2^S in terms of ordering by inclusion of graphs.

THEOREM 1. *If the nonlinear operator* $N: S \to S$, *defined by* $Nu = -g(x, y, u)$, *is such that*

(i) N *is continuous,*

(ii) $\langle Nu - Nv, u - v \rangle \geq 0 \quad (u, v \in S)$,

then the nonlinear problem (1) *has a unique solution.*

Proof. It follows from (4) that the operator $-H(I - P)$ is linear and monotone. Hence, by a result of P. Hess [8], the auxiliary equation (5) has a unique solution $u \in S$ for each $x^* \in S_0$. We now consider the bifurcation equation (6). First we show that the operator

$$T = PN[I - H(I - P)N]^{-1}: S_0 \to S_0$$

is maximal monotone. Let x^* and y^* be any two elements of S_0, and let u and v be the corresponding unique solutions of the auxiliary equation. Then

$$u - H(I - P)Nu = x^* \quad \text{and} \quad v - H(I - P)Nv = y^*.$$

Proceeding as in [12], and using the fact that N and $-H(I - P)$ are monotone, we see that

$$\langle Tx^* - Ty^*, x^* - y^* \rangle = \langle PNu - PNv, x^* - y^* \rangle = \langle Nu - Nv, x^* - y^* \rangle$$
$$= \langle Nu - Nv, u - v \rangle + \langle Nu - Nv, -H(I - P)(Nu - Nv) \rangle \geq 0.$$

Thus $T: S_0 \to S_0$ is monotone. In order to show that T is maximal monotone, it suffices (by the theorem of G. J. Minty [13]) to show that the equation $x + Tx = x^*$ has a solution $x \in S_0$ for each $x^* \in S_0$. By the theorem of Hess [11] quoted above, the equation $u - H(I - P)Nu + PNu = x^*$ has a unique solution $u \in S$ for each $x^* \in S_0$. Hence

$$u - H(I - P)Nu = x^* - PNu \in S_0.$$

Let $y^* = u - H(I - P)Nu$. Then

$$u - H(I - P)Nu + PNu = x^* \quad \text{implies } y^* + PN[I - H(I - P)N]^{-1} y^* = x^*,$$

that is, y^* is a solution of the equation $x + Tx = x^*$. Hence T is maximal monotone.

Finally, since the eigenvalues λ_i are such that $0 < \lambda_1 < \lambda_2 < \cdots$, it follows that $-\Delta$ is maximal monotone over S_0 and $\langle -\Delta x^*, x^* \rangle \geq \lambda_1 \|x^*\|^2$. Hence $T - \Delta$ is maximal monotone and coercive over S_0. Thus the bifurcation equation (6) is uniquely solvable. This completes the proof.

We now apply the ideas in the proof of Theorem 1 to the case where $-N$ is monotone.

THEOREM 2. *Let* $N: S \to S$ $(Nu = -g(x, y, u))$ *be such that*

(i) N *is continuous and bounded,*

(ii) *there exists* $p > 0$ *such that* $p < \lambda_{m+1}$ *and*

$$\langle Nu_1 - Nu_2, u_1 - u_2 \rangle \geq -p \|u_1 - u_2\|^2 \quad \textit{for all } u_1, u_2 \in S,$$

(iii) *there exists* $q > \lambda_m$ *such that* $\langle Nu_1 - Nu_2, u_1^* - u_2^* \rangle \leq -q \|u_1^* - u_2^*\|^2$ *whenever* $u_1^*, u_2^* \in S_0$, $Pu_1 = u_1^*$, *and* $Pu_2 = u_2^*$.

Then the nonlinear problem (1) *has a unique solution.*

Proof. Since the operator N is continuous and bounded and $-H(I - P)$ is linear and compact, it follows that $-H(I - P)N$ is compact. In order to solve the auxiliary equation (5), we use the following variant of the Schauder principle of invariance of domain (see M. Nagumo [14]): If $T: S \to S$ is compact, $I + T$ is one-to-one, and $(I + T)^{-1}$ is bounded, then the equation $u + Tu = v$ has a unique solution for each $v \in S$. By the methods of [9, Proposition 1] it can be shown that the auxiliary equation (5) has a unique solution for each $x^* \in S_0$ and the operator $[I - H(I - P)N]^{-1}$ is continuous and bounded. The bifurcation equation (6) can be rewritten as

$$x^* + P[PN(I - H(I - P)N)^{-1} - P - \Delta]x^* = 0; \tag{7}$$

this is again of the form $(I + T)x^* = 0$, where T is compact. A repeated application of the Schauder principle as stated above gives a unique solution for the bifurcation equation. This proves the theorem.

We now extend the ideas of the earlier theorems to the case of nonlinear perturbations at resonance. As we remarked in the introduction, the first result in this direction for elliptic boundary value problems was obtained by Landesman and Lazer [11], who generalized the earlier results of Lazer and Leach [12] for ordinary differential equations. Thus, for example, they consider the nonlinear problem over $A_1 = [0, 1] \times [0, 1]$,

$$\Delta u + \lambda u + g(u) = 0 \quad \text{in } A_1,$$

$$u = 0 \quad \text{on } \partial A_1,$$

where $\lim_{s \to \infty} g(s) = g(-\infty) \leq g(u) \leq g(+\infty) = \lim_{s \to \infty} g(s)$, the two limits being assumed to be finite (for example $g(u) = \arctan u$).

THEOREM 3. *Let the nonlinear operator* $M: S \to S$ *defined by* $Mu = h(x, y, u)$ *be such that*

(i) M *is continuous and bounded,*

(ii) $\langle Mu_1 - Mu_2, u_1 - u_2 \rangle \leq \mu \|u_1 - u_2\|^2$ *for some* $\mu < \lambda_{m+1} - \lambda_m$,

(iii) *there exists* $R > 0$ *such that* $\langle Mu, x^* \rangle \geq 0$ *for each* x^* *satisfying the conditions* $\|x^*\| = R$ *and* $Pu = x^*$.

Then the nonlinear problem

$$(8) \qquad \begin{cases} \Delta u + \lambda_m u + h(x, y, u) = 0 & ((x, y) \in A), \\ u = 0 & ((x, y) \in \partial A) \end{cases}$$

has at least one solution.

Proof. It is easy to see that the operator $N: S \to S$ defined by

$$Nu = -\lambda_m u(x, y) - h(x, y, u)$$

satisfies hypothesis (ii) of Theorem 2; thus the auxiliary equation corresponding to (8) is uniquely solvable for each $x^* \in S_0$, and the operator $(I - H(I - P)N)^{-1}$ is continuous and bounded. We now consider the solvability of the bifurcation equation (7), and we apply the following result from [11]: Let T be a continuous and bounded mapping of S_0 into itself. Also, let there exist $R > 0$ such that

$$\langle Tx^*, x^* \rangle \geq -\|x^*\|^2$$

for all x^* satisfying the condition $\|x^*\| = R$. Then the equation $(I + T)x^* = 0$ has at least one solution.

Now, in our situation it follows from (7) that $T = \Delta - PN[I - H(I - P)N]^{-1} - P$. Thus

$$\langle Tx^*, x^* \rangle = \langle \Delta x^*, x^* \rangle - \langle PN[I - H(I - P)N]^{-1}x^*, x^* \rangle - \langle Px^*, x^* \rangle$$
$$\geq -\lambda_m \|x^*\|^2 - \|x^*\|^2 - \langle PNu, x^* \rangle,$$

where u is the unique solution of the auxiliary equation corresponding to x^*. Thus

$$\langle Tx^*, x^* \rangle \geq -\lambda_m \|x^*\|^2 - \|x^*\|^2 - \langle Nu, x^* \rangle.$$

But $Nu = -\lambda_m u - h(x, y, u)$. Thus

$$\langle Tx^*, x^* \rangle \geq -\lambda_m \|x^*\|^2 - \|x^*\|^2 + \lambda_m \langle u, x^* \rangle + \langle h(x, y, u), x^* \rangle$$
$$= -\|x^*\|^2 + \langle h(x, y, u), x^* \rangle.$$

Hence, if there exists $R > 0$ such that $\langle h(x, y, u), x^* \rangle \geq 0$ for all $\|x^*\| = R$, then $\langle Tx^*, x^* \rangle \geq -\|x^*\|^2$; therefore the bifurcation equation (7) is solvable.

Remark. In a forthcoming paper [11], existence for nonlinear Hammerstein equations of the type (7) have been obtained when the linear operator $-H(I - P)$ possesses a decomposition of the type J^*J and the nonlinear operator N is such that $D(N) \supset D(J^*)$. In view of these results, it is possible to extend the methods of this paper to obtain existence results for problem (1) where g involves derivatives of u.

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