COMPACT FAMILIES OF UNIVALENT FUNCTIONS AND THEIR SUPPORT POINTS

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1. INTRODUCTION

Let D be a plane domain and H(D) the space of analytic functions on D endowed with the topology of locally uniform convergence. It is well known that H(D) is a metrizable, locally convex topological vector space; we denote by H'(D) its topological dual space and by $H_u(D)$ the set of univalent functions in H(D). We shall be interested in subsets of $H_u(D)$ whose elements are normalized by two continuous linear functionals, that is, in subsets of the form

(1) $$\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q) = \{f \in H_u(D): \ell_1(f) = P,\ \ell_2(f) = Q\},$$

where ℓ_1 and ℓ_2 denote fixed functionals in H'(D) and P and Q denote points in **C**.

A number of the standard families of univalent functions are of the form (1). We single out two:

(i) Let $z_0 \in D$ and $\ell_1(f) = f(z_0)$, $\ell_2(f) = f'(z_0)$, $P = 0$, $Q = 1$. Then

(2) $$\mathscr{S}(D, z_0) = \{f \in H_u(D): f(z_0) = 0,\ f'(z_0) = 1\}$$

is of the form (1). For $U = \{z: |z| < 1\}$, the set $S = \mathscr{S}(U, 0)$ is the familiar normalized schlicht class.

(ii) Let $p, q \in D$ $(p \neq q)$, $P, Q \in \mathbf{C}$ $(P \neq Q)$, and $\ell_1(f) = f(p)$, $\ell_2(f) = f(q)$. Then the family

(3) $$\mathscr{T}(D, p, q, P, Q) = \{f \in H_u(D): f(p) = P,\ f(q) = Q\},$$

normalized at two points, is of the form (1).

In order to solve extremal problems over such families $\mathscr{F}$, it is useful to know whether $\mathscr{F}$ is compact. Our first result (Theorem 1) is a characterization of the nonempty and compact families $\mathscr{F}$ in terms of the normalizing functionals ℓ_1 and ℓ_2 and the constants P and Q.

A function $f \in \mathscr{F}$ is said to be a *support point* of $\mathscr{F}$ if and only if $\Re L(f) = \sup_{\mathscr{F}} \Re L$, for some $L \in H'(D)$ that is not constant on $\mathscr{F}$. Geometrically, at a support point the family $\mathscr{F}$ has a supporting hyperplane. Theorems 2 and 3 (and 3') concern the mapping properties of support points for compact families $\mathscr{F}$. Applications to the families $\mathscr{S}(D, z_0)$ and $\mathscr{T}(D, p, q, P, Q)$ are contained in Theorems 4 and 5 (and 4' and 5').

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2. REPRESENTING MEASURES FOR $\ell \in H'(D)$

If $\ell \in H'(D)$, then there exists a finite complex Borel measure μ with compact support $K \subset D$ such that

$$\ell(f) = \int_K f\, d\mu \qquad \text{for each } f \in H(D). \tag{4}$$

Since the support of μ is precisely the subset K of D, the integral $\int_K f\, d\mu$ can be used to extend the definition of $\ell(f)$ to functions f integrable on K. Then, for example, the formulas

$$\ell\left(\frac{1}{\zeta - z}\right) = \int_K \frac{d\mu(\zeta)}{\zeta - z} \quad \text{and} \quad \ell\left(\frac{1}{f - w}\right) = \int_K \frac{d\mu(\zeta)}{f(\zeta) - w} \quad (f \in H(D)) \tag{5}$$

define analytic functions of $z \in \mathbf{C} - K$ and $w \in \mathbf{C} - f(K)$, vanishing at ∞. We shall use the same symbol for $\ell \in H'(D)$ and its extension to functions integrable on K.

The measure μ (and support K) representing $\ell \in H'(D)$ is not unique, but it has the following property.

PROPOSITION 1. *Suppose* $\ell \in H'(D)$ *and* K *is a compact set in* D *containing the support of a representing measure for* ℓ. *If* $\ell\left(\frac{1}{\zeta - z}\right) = 0$ *for each* $z \in \mathbf{C} - K$, *then* $\ell \equiv 0$ *on* H(D).

For a reference to these well-known facts, see for example [6, p. 377] or [3, p. 159]. An elementary consequence of Proposition 1 is the following.

PROPOSITION 2. *Suppose* $\ell \in H'(D)$ *and* K *is a compact set in* D *containing the support of a representing measure for* ℓ. *If* $g \in H_u(D)$ *and* $\ell\left(\frac{1}{g - w}\right) = 0$ *for each* $w \in \mathbf{C} - g(K)$, *then* $\ell \equiv 0$ *on* H(D).

Proof. If μ is a representing measure for $\ell \in H'(D)$ with support in a compact set $K \subset D$ and $g \in H_u(D)$, then $\tilde{\mu} = \mu \circ g^{-1}$ is a measure supported in the compact set g(K), and

$$\tilde{\ell}(\tilde{f}) = \int_{g(K)} \tilde{f}\, d\tilde{\mu}$$

defines a continuous linear functional on H(g(D)). If $\tilde{\ell}\left(\frac{1}{\omega - w}\right) = 0$ for each $w \in \mathbf{C} - g(K)$, then $\tilde{\ell}(\tilde{f}) = 0$ for each $\tilde{f} \in H(g(D))$, by Proposition 1. The assertion now follows by a change of variables.

3. COMPACT FAMILIES

It is convenient to associate with $\ell_1, \ell_2 \in H'(D)$ and $P, Q \in \mathbf{C}$ two new functionals

$$\ell_0 = \frac{1}{\ell_1(Q) - \ell_2(P)} [\ell_2(1)\, \ell_1 - \ell_1(1)\, \ell_2], \tag{6a}$$

(6b) $$\tilde{\ell}_0 = \frac{1}{\ell_1(Q) - \ell_2(P)} [Q \cdot \ell_1 - P \cdot \ell_2],$$

in case $\ell_1(Q) \neq \ell_2(P)$. Observe that we do not distinguish between the constants P, Q, 1 and the corresponding constant functions. Note that

(7) $$\ell_0(1) = 0, \quad \tilde{\ell}_0(1) = 1,$$

and that for $f \in \mathscr{F}(D, \ell_1, \ell_2, P, Q)$,

(8) $$\ell_0(f) = -1, \quad \tilde{\ell}_0(f) = 0.$$

It is an elementary exercise to verify that if $g \in H_u(D)$, then

(9) $$T(g) = \frac{-1}{\ell_0(g)} [g - \tilde{\ell}_0(g)] \in \mathscr{F}(F, \ell_1, \ell_2, P, Q).$$

Definition. A domain D has a *strongly dense boundary* if for each g in $H_u(D)$ the only degenerate (one-point) components of $\bar{C} - g(D)$ are cluster points of non-degenerate components.

Examples of domains with strongly dense boundaries are all simply connected domains with at least two boundary points and all finitely connected domains without degenerate boundary components (in $\bar{C}$). In these examples, $\bar{C} - g(D)$ has no degenerate components, for each $g \in H_u(D)$.

THEOREM 1. *Let* $\ell_1, \ell_2 \in H'(D)$ *and* $P, Q \in \mathbf{C}$. *If*

(a) $\ell_1(Q) \neq \ell_2(P)$

and

(b) $\ell_2(1)\,\ell_1(g) \neq \ell_1(1)\,\ell_2(g)$ *for each* $g \in H_u(D)$, *then the family* $\mathscr{F}(D, \ell_1, \ell_2, P, Q)$ *is nonempty and compact.*

Conversely, if $\mathscr{F}(D, \ell_1, \ell_2, P, Q)$ *is nonempty and compact, then* (a) *holds. If, in addition,* D *has a strongly dense boundary, then* (b) *holds.*

COROLLARY. *Suppose that* D *is a finitely connected domain without degenerate boundary components* (*in* $\bar{C}$), *that* $\ell_1, \ell_2 \in H'(D)$, *and that* $P, Q \in \mathbf{C}$. *Then* $\mathscr{F}(D, \ell_1, \ell_2, P, Q)$ *is nonempty and compact if and only if* (a) *and* (b) *hold.*

Proof of Theorem 1. The compactness of the set $\mathscr{S}(D, z_0)$ (defined in (2)) is well known [4]. Indeed, locally uniform boundedness follows if we apply Bieberbach's distortion theorem in each simply connected subdomain of D containing z_0. Also, $\mathscr{S}(D, z_0)$ is nonempty since $(z - z_0) \in \mathscr{S}(D, z_0)$.

If (a) and (b) hold, then the associated functionals ℓ_0 and $\tilde{\ell}_0$ are defined, and $\ell_0(g) \neq 0$ for each $g \in H_u(D)$. Therefore the transform T defined in (9) is continuous on $H_u(D)$ and maps $\mathscr{S}(D, z_0)$ onto a nonempty compact subset of $\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q)$. If $f \in \mathscr{F}$, then $g = \frac{f - f(z_0)}{f'(z_0)} \in \mathscr{S}(D, z_0)$, and an elementary calculation shows that $T(g) = f$. Therefore $T(\mathscr{S}(D, z_0)) = \mathscr{F}$ is nonempty and compact.

Conversely, assume $\mathscr{F} \neq \emptyset$ and fix an $f \in \mathscr{F}$. If (a) does not hold, then the linear system

$$\ell_1(Af + B) = A \cdot P + B \cdot \ell_1(1) = P,$$

$$\ell_2(Af + B) = A \cdot Q + B \cdot \ell_2(1) = Q$$

has rank at most one and is consistent, since $A = 1$, $B = 0$ is a solution. Therefore $\mathscr{F}$ cannot be compact.

Suppose now $\mathscr{F} \neq \emptyset$, D has a strongly dense boundary, and (a) holds, but not (b). Then there exists a $g \in H_u(D)$ such that $\ell_0(g) = 0$. Let K be the support of a representing measure for ℓ_0. If $\ell_0\left(\frac{1}{g - w}\right) = 0$ for each $w \in \mathbf{C} - g(D)$, then since $\bar{\mathbf{C}} - g(D)$ has no isolated points, we see that $\ell_0\left(\frac{1}{g - w}\right) \equiv 0$ for each w in some open neighborhood N of $\bar{\mathbf{C}} - g(D)$ $(N \subset \bar{\mathbf{C}} - g(K))$. Now $D - g^{-1}(N)$ is a compact subset of D containing K. By Proposition 2, $\ell_0 \equiv 0$. This, however, is not the case since ℓ_0 is -1 on $\mathscr{F}$. Therefore there exists a $w_0 \in \mathbf{C} - g(D)$ such that $\ell_0\left(\frac{1}{g - w_0}\right) \neq 0$.

If w_0 belongs to a degenerate component of $\mathbb{C} - g(D)$, then by the continuity of $\ell_0\left(\frac{1}{g - w}\right)$ and the strongly-dense-boundary property, we may replace it by a point on a nondegenerate component of $\mathbf{C} - g(D)$. In any case, there is now a point w_0 on a nondegenerate component of $\mathbf{C} - g(D)$ such that $\ell_0\left(\frac{1}{g - w_0}\right) \neq 0$.

By composing g with the Schiffer boundary variations [8], we obtain functions $g_n \in H_u(D)$ of the form

$$g_n = g + \frac{\varepsilon_n}{g - w_0} + o(\varepsilon_n),$$

where $\varepsilon_n \to 0$ and the term $o(\varepsilon_n)/\varepsilon_n$ converges to zero uniformly on compact subsets of D as $n \to \infty$. Since $\ell_0(g) = 0$, we have the relation

$$\ell_0(g_n) = \varepsilon_n \ell_0\left(\frac{1}{g - w_0}\right) + o(\varepsilon_n) \neq 0$$

for all sufficiently large n. At the same time, each of the functions

$$f_n = T(g_n) = \frac{-1}{\ell_0(g_n)}[g_n - \tilde{\ell}_0(g_n)] = \frac{-1}{\varepsilon_n \ell_0\left(\frac{1}{g - w_0}\right)}[g - \tilde{\ell}_0(g)] + o(1)$$

belongs to $\mathscr{F}$, but $\{f_n\}$ has no convergent subsequence. Therefore $\mathscr{F}$ is not compact.

Remarks. If we know a priori that $\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q) \neq \emptyset$, then we can assert that (b) implies compactness. Indeed, (a) follows from (b) if we substitute an $f \in \mathscr{F}$.

That (a) alone is not sufficient for compactness is evident from the family

$$\mathscr{G} = \{g \in H_u(U): g(0) = 0,\ g''(0) = 2\},$$

which satisfies (a), but is not compact since $nz + z^2 \in \mathscr{G}$ for every $n \geq 2$.

In the following sections we shall use as examples the families (2) and (3). Both are well known to be compact and, in fact, the former was used in the proof of Theorem 1. However, it is instructive to see how Theorem 1 applies in these two cases.

(i) For $\mathscr{S}(D, z_0)$, (a) is the statement that $1 \neq 0$ and (b) that $0 \neq g'(z_0)$ for each $g \in H_u(D)$. Both are obviously true.

(ii) For $\mathscr{T}(D, p, q, P, Q)$, (a) is the statement that $Q \neq P$ and (b) that $g(p) \neq g(q)$ for each $g \in H_u(D)$. Again both are obviously true.

Another familiar compact family of the form (1) is the family

$$\Sigma' = \left\{ f \in H_u(0 < |z| < 1): f(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \right\}.$$

For this family one easily verifies that condition (a) is satisfied, but that condition (b) is violated. The family Σ' is not a counterexample to the necessity part of Theorem 1, but rather shows that the assumption of a strongly dense boundary is not superfluous.

4. THE FUNCTIONS $U(w; f)$ AND $\tilde{U}(w; f)$

Suppose $\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q)$ is nonempty and compact. Then, by Theorem 1, the functionals ℓ_0 and $\tilde{\ell}_0$ in (6) are defined. Let K and $\tilde{K}$ be the supports of representing measures for ℓ_0 and $\tilde{\ell}_0$, respectively. Then, following (5), we may consider for each $f \in H(D)$ the functions

$$U(w; f) = \ell_0\left(\frac{1}{f - w}\right) \quad \text{and} \quad \tilde{U}(w; f) = \tilde{\ell}_0\left(\frac{1}{f - w}\right) \tag{10}$$

to an analytic in $\Omega = \mathbf{C} - f(K)$ and $\tilde{\Omega} = \mathbf{C} - f(\tilde{K})$, respectively.

Writing

$$\frac{1}{f - w} = -\frac{1}{w} - \frac{f}{w^2} - \frac{f^2}{1 - \frac{1}{w} f} \cdot \frac{1}{w^3},$$

we see from (7) and (8) that

$$U(w; f) = \frac{1}{w^2} + O\left(\frac{1}{w^3}\right) \quad \text{and} \quad \tilde{U}(w; f) = -\frac{1}{w} + O\left(\frac{1}{w^3}\right) \tag{11}$$

as $w \to \infty$, for $f \in \mathscr{F}$.

We consider our examples:

(i) For $f \in \mathscr{S}(D, z_0)$, we have the functions

$$U(w; f) = \frac{1}{w^2} \quad \text{and} \quad \tilde{U}(w; f) = \frac{-1}{w} \quad \text{for } w \in \Omega = \tilde{\Omega} = \mathbf{C} - \{0\}. \tag{12}$$

We note for later reference that the trajectories of the vector field $\operatorname{grad}\left[\Re\int\sqrt{U(w;f)}\,dw\right]$ are the rays emanating from the origin.

(ii) For $f \in \mathscr{F}(D, p, q, P, Q)$, we have the functions

$$U(w;f) = \frac{1}{(w-P)(w-Q)} \quad \text{and} \quad \tilde{U}(w;f) = \frac{P+Q-w}{(w-P)(w-Q)} \tag{13}$$

$$\text{for } w \in \Omega = \tilde{\Omega} = \mathbf{C} - \{P, Q\}\ .$$

For future use we observe that the trajectories of the vector field $\operatorname{grad}\left[\Re\int\sqrt{U(w;f)}\,dw\right]$ are the hyperbolae with foci P and Q.

It will be important to know that $U(w;f)$ does not vanish on nondegenerate components of $\mathbf{C} - f(D)$.

LEMMA 1. *Assume* $\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q)$ *is compact and* $f \in \mathscr{F}$.

(a) *If* D *has a strongly dense boundary, then* $U(w;f) \neq 0$ *for each* $w \in \mathbf{C} - f(D)$.

(b) *If* γ *is a nondegenerate component of* $\mathbf{C} - f(D)$, *then either* $U(w;f) \neq 0$ *for each* $w \in \gamma$ *or* $U(w;f) \equiv 0$ *on* γ.

(c) *If* γ *is an unbounded nondegenerate component of* $\dot{\mathbf{C}} - f(D)$, *then* $U(w;f) \neq 0$ *for each* $w \in \gamma$.

(d) *If* γ *is a nondegenerate component of* $\mathbf{C} - f(D)$ *and the support of some representing measure for* ℓ_0 *does not separate* γ *from* ∞, *then* $U(w;f) \neq 0$ *for each* $w \in \gamma$.

Proof. (a): If D has a strongly dense boundary, then

$$U(w;f) = \ell_0\left(\frac{1}{f-w}\right) \neq 0$$

for each $w \in \mathbf{C} - f(D)$, by Theorem 1 and since $\frac{1}{f-w} \in H_u(D)$.

(b): Suppose $U(w_0;f) = 0$ for some $w_0 \in \gamma$, but $U(w;f) \not\equiv 0$ on γ. Then for some $\varepsilon > 0$ we can assert that $U(w;f) \neq 0$ whenever $0 < |w - w_0| < \varepsilon$. The inversion $t = \frac{1}{w - w_0}$ maps γ onto a nondegenerate continuum Γ containing ∞, and

$$U\left(w_0 + \frac{1}{t}; f\right) \neq 0 \qquad \text{for } \frac{1}{\varepsilon} < |t| < \infty .$$

Let $g = \frac{1}{f - w_0}$. Then $g \in H_u(D)$ and $\ell_0(g) = U(w_0;f) = 0$. For

$$t_1 \in \Gamma \cap \left\{\frac{1}{\varepsilon} < |t| < \infty\right\}$$

we have the relations

$$\ell_0\left(\frac{1}{g - t_1}\right) = -\frac{1}{t_1}\,\ell_0(1) - \frac{1}{t_1^2}\,\ell_0\left(\frac{1}{f - (w_0 + 1/t_1)}\right) = -\frac{1}{t_1^2}\,U\left(w_0 + \frac{1}{t_1};\, f\right) \neq 0.$$

Therefore, from the function g we may construct variations $g_n = g + \dfrac{\varepsilon_n}{g - t_1} + o(\varepsilon_n)$ such that $f_n = T(g_n) \in \mathcal{F}$ and the sequence $\{f_n\}$ contradicts the compactness of $\mathcal{F}$, just as in the proof of Theorem 1.

(c) is a special case of (d).

(d): From (11) we see that the analytic function $U(w;\, f) = \ell_0\left(\dfrac{1}{f - w}\right)$ has a zero of finite order at ∞. Since the support of some representing measure for ℓ_0 does not separate γ from ∞, the function $U(w;\, f)$ does not vanish identically on γ. The result now follows from part (b).

5. AN ELEMENTARY VARIATION

In this section we begin to study the problem

$$\max_{\mathcal{F}(D,\ell_1,\ell_2,P,Q)} \Re\, L,$$

where $L \in H'(D)$. It will be convenient to associate with $L \in H'(D)$ and $f \in \mathcal{F}(D, \ell_1, \ell_2, P, Q)$ the functional

(14) $$L_f = L + L(f)\,\ell_0 - L(1)\,\tilde{\ell}_0.$$

Observe that

$$L_f\left(\frac{1}{f - w}\right) = L\left(\frac{1}{f - w}\right) + U(w;\, f)\, L(f) - \tilde{U}(w;\, f)\, L(1).$$

LEMMA 2. *Suppose* $\mathcal{F} = \mathcal{F}(D, \ell_1, \ell_2, P, Q)$ *is compact,* $f \in \mathcal{F}$, *and* $L \in H'(D)$. *If* $\Re\, L(f) = \max_{\mathcal{F}} \Re\, L$, *then*

(15) $$\Re\left\{\frac{1}{U(w;\, f)}\, L_f\left(\frac{1}{f - w}\right)\right\} \geq 0$$

for each $w \in \mathbf{C} - f(D)$ *such that* $U(w;\, f) \neq 0$.

Proof. Fix $f \in \mathcal{F}$ and $w \in \mathbf{C} - f(D)$, and assume $U(w;\, f) \neq 0$. Then

$$g = \frac{1}{f - w} \in H_u(D) \quad \text{and} \quad T(g) = \frac{-1}{U(w;\, f)}\left[\frac{1}{f - w} - \tilde{U}(w;\, f)\right] \in \mathcal{F}.$$

If $\Re\, L(f) = \max_{\mathcal{F}} \Re\, L$, then

$$0 \leq \Re\, L(f - T(g)) = \Re\left\{\frac{1}{U(w;\, f)}\, L_f\left(\frac{1}{f - w}\right)\right\}.$$

The condition (15) carries information only if $L_f\left(\dfrac{1}{f - w}\right) \not\equiv 0$ on $\mathbf{C} - f(D)$. The next two lemmas show that the latter is the case when L is not trivial on $\mathcal{F}$.

LEMMA 3. *Suppose* $\bar{C} - D$ *has no isolated points,* $\mathcal{F} = \mathcal{F}(D, \ell_1, \ell_2, P, Q)$ *is compact, and* $f \in \mathcal{F}$. *If* $L \in H'(D)$ *is linearly independent of* ℓ_1 *and* ℓ_2 *(for example, if* $L \not\equiv$ constant *on* $\mathcal{F}$*), then*

$$L_f\left(\frac{1}{f - w}\right) \not\equiv 0 \qquad \textit{on } C - f(D).$$

Proof. If $L_f\left(\frac{1}{f - w}\right) \equiv 0$ on $C - f(D)$, then, since $C - f(D)$ has no isolated points, $L_f\left(\frac{1}{f - w}\right) \equiv 0$ on an open neighborhood of $\bar{C} - f(D)$. From Proposition 2 it follows that $L_f \equiv 0$ on $H(D)$. Therefore L is a linear combination of ℓ_0 and $\tilde{\ell}_0$, hence of ℓ_1 and ℓ_2.

The conclusion of Lemma 3 still allows $L_f\left(\frac{1}{f - w}\right)$ to vanish identically on some nondegenerate components. In the following lemma, we eliminate this possibility.

LEMMA 4. *Suppose* $\mathcal{F} = \mathcal{F}(D, \ell_1, \ell_2, P, Q)$ *is compact,* $f \in \mathcal{F}$, *and* $L \in H'(D)$ *is linearly independent of* ℓ_1 *and* ℓ_2. *If the support of some representing measure for the functional* L_f *does not separate the components of* $\bar{C} - D$, *then*

$$L_f\left(\frac{1}{f - w}\right) \not\equiv 0$$

on each nondegenerate component of $C - f(D)$.

Proof. If $L_f\left(\frac{1}{f - w}\right) \equiv 0$ on a nondegenerate component of $C - f(D)$, then $L_f\left(\frac{1}{f - w}\right) \equiv 0$ on $C - f(K)$, where K supports a representing measure for L_f and does not separate the components of $\bar{C} - D$. Again by Proposition 2, $L_f \equiv 0$ on $H(D)$, and L is a linear combination of ℓ_1 and ℓ_2.

6. SCHIFFER'S BOUNDARY VARIATION

Let f belong to a compact family $\mathcal{F} = \mathcal{F}(D, \ell_1, \ell_2, P, Q)$, and let w be a point of a nondegenerate component γ of $C - f(D)$. Then, according to a theorem of M. Schiffer [8], there exist variations of f of the form

$$g = f + \frac{\varepsilon}{f - w} + o(\varepsilon)$$

that are univalent in D. We may normalize these variations by the transformation

$$f^* = T(g) = \frac{-1}{\ell_0(g)}\,[g - \tilde{\ell}_0(g)] = f + \varepsilon\left[\frac{1}{f - w} + U(w; f)\cdot f - \tilde{U}(w; f)\right] + o(\varepsilon),$$

so that $f^* \in \mathcal{F}$. If now $L \in H'(D)$ and $\Re L(f) = \max_{\mathcal{F}} \Re L$, then

$$\Re L(f^* - f) = \Re\left\{\varepsilon L_f\left(\frac{1}{f - w}\right)\right\} + o(\varepsilon) \leq 0.$$

If the analytic function $L_f\left(\frac{1}{f - w}\right)$ does not vanish identically on γ, then by Schiffer's fundamental lemma [8], γ consists of finitely many analytic arcs satisfying the differential equation

$$L_f\left(\frac{1}{f - w}\right)(dw)^2 > 0. \tag{16}$$

In fact, if $L_f\left(\frac{1}{f - w}\right) \not\equiv 0$ on γ, then γ is a single analytic arc:

LEMMA 5. *Suppose* $\mathcal{F} = \mathcal{F}(D, \ell_1, \ell_2, P, Q)$ *is compact*, $f \in \mathcal{F}$, $L \in H'(D)$, *and* $\Re L(f) = \max_{\mathcal{F}} \Re L$. *If* γ *is a nondegenerate component of* $\mathbf{C} - f(D)$ *on which* $L_f\left(\frac{1}{f - w}\right) \not\equiv 0$ *and* $U(w; f) \not\equiv 0$, *then* γ *is a single analytic arc. Moreover,* γ *satisfies the differential equation* $L_f\left(\frac{1}{f - w}\right)(dw)^2 > 0$ *as long as* $L_f\left(\frac{1}{f - w}\right) \neq 0$. *Furthermore, if* $L_f\left(\frac{1}{f - w}\right)$ *has a zero* $w_0 \in \gamma$, *then* γ *lies on the straight line* $\left\{w_0 + \frac{t}{\sqrt{U(w_0; f)}} : t \in (-\infty, \infty)\right\}$.

Remark. The functional L_f is defined in (14). Criteria under which $L_f\left(\frac{1}{f - w}\right) \not\equiv 0$ are given in Lemmas 3 and 4. Similarly, criteria under which $U(w; f) \not\equiv 0$ are given in Lemma 1.

Proof. Let $\mathcal{F}$, f, L, γ, L_f, U be as in the hypothesis. By Schiffer's fundamental lemma, the only possible points of nonanalyticity or branching of γ are points where $L_f\left(\frac{1}{f - w}\right)$ vanishes. However, we shall show that if $L_f\left(\frac{1}{f - w}\right)$ has a zero, then γ lies on a straight line, hence is an analytic arc.

Assume therefore that $L_f\left(\frac{1}{f - w_0}\right) = 0$ for some $w_0 \in \gamma$. Then $L_f\left(\frac{1}{f - w}\right) \neq 0$ for all $w \neq w_0$ in a neighborhood of w_0 since $L_f\left(\frac{1}{f - w}\right) \not\equiv 0$ on γ. Furthermore, since $U(w; f) \not\equiv 0$ on γ, by Lemma 1(b) the function $U(w; f)$ never vanishes on γ. Since $w_0 \in \gamma$, the function $\frac{1}{f - w_0}$ belongs to $H_u(D)$ and

$$\hat{f} = T\left(\frac{1}{f - w_0}\right) = \frac{-1}{U(w_0; f)}\left[\frac{1}{f - w_0} - \tilde{U}(w_0; f)\right]$$

belongs to $\mathcal{F}$. At the same time, the mapping

$$\hat{w} = \hat{w}(w) = \frac{-1}{U(w_0; f)}\left[\frac{1}{w - w_0} - \tilde{U}(w_0; f)\right]$$

takes γ onto a nondegenerate continuum $\hat{\gamma} \subset \overline{\mathbf{C}} - \hat{f}(D)$ containing $\infty = \hat{w}(w_0)$. We note for future use that

$$\frac{d\hat{w}}{dw} = \frac{1}{U(w_0; f)(w - w_0)^2}. \tag{17}$$

Observe that

$$L(\hat{f}) = \frac{-1}{U(w_0; f)}\left[L\left(\frac{1}{f - w_0}\right) - \tilde{U}(w_0; f)\, L(1) \right]$$
$$= \frac{-1}{U(w_0; f)}\left[L_f\left(\frac{1}{f - w_0}\right) - L(f)\, U(w_0; f) \right] = L(f),$$

so that $\hat{f}$ is also an extremal function for the problem $\max_{\mathscr{F}} \Re L$. Since $L(\hat{f}) = L(f)$, the functionals $L_{\hat{f}}$ and L_f are identical, and by direct computation

$$L_{\hat{f}}\left(\frac{1}{\hat{f} - \hat{w}}\right) = L_f\left(\frac{1}{\hat{f} - \hat{w}}\right) = (w - w_0)\, U(w_0; f)\, L_f\left(\frac{f - w_0}{f - w}\right)$$

(18)

$$= (w - w_0)^2\, U(w_0; f)\, L_f\left(\frac{1}{f - w}\right),$$

since $L_f(1) = 0$. Now, since $\hat{f}$ is extremal and $L_{\hat{f}}\left(\frac{1}{\hat{f} - \hat{w}}\right) \not\equiv 0$ on $\hat{\gamma}$, we may again apply Schiffer's fundamental lemma [8], this time to $\hat{\gamma}$, to learn that $\hat{\gamma}$ consists of finitely many analytic arcs satisfying the differential equation

$$L_{\hat{f}}\left(\frac{1}{\hat{f} - \hat{w}}\right)(d\hat{w})^2 > 0. \tag{19}$$

For all $w \in \gamma$ $(w \neq w_0)$ in a sufficiently small neighborhood of w_0, the quotient of (19) and (16) together with the relations (18) and (17), yields the relations

$$0 < \frac{L_{\hat{f}}\left(\frac{1}{\hat{f} - \hat{w}}\right)(d\hat{w})^2}{L_f\left(\frac{1}{f - w}\right)(dw)^2} = (w - w_0)^2\, U(w_0; f)\left(\frac{d\hat{w}}{dw}\right)^2 = \frac{1}{U(w_0; f)\,(w - w_0)^2}.$$

That is, w must be on the straight line $\left\{ w_0 + \frac{t}{\sqrt{U(w_0; f)}} : t \in (-\infty, \infty) \right\}$. In particular, γ is an analytic arc in a neighborhood of w_0.

We have shown in any case that γ is a single analytic arc. Furthermore, if $L_f\left(\frac{1}{f - w}\right)$ has a zero on γ, then γ lies locally, and hence globally (by its analyticity), on the indicated line.

7. THE $\pi/4$-THEOREM

THEOREM 2. *Suppose* $\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q)$ *is compact,* $f \in \mathscr{F}$, $L \in H'(D)$, *and* $\Re L(f) = \max_{\mathscr{F}} \Re L$. *If* γ *is a nondegenerate component of* $\mathbf{C} - f(D)$ *on which* $L_f\left(\frac{1}{f - w}\right) \not\equiv 0$ *and* $U(w; f) \not\equiv 0$, *then* γ *is an analytic arc whose tangent makes an angle at most* $\pi/4$ *with respect to the vector field* $\operatorname{grad}\left[\Re \int \sqrt{U(w; f)}\, dw \right]$.

Proof. The analyticity of γ is a consequence of Lemma 5. Fix $w_0 \in \gamma$. If $L_f\left(\frac{1}{f - w_0}\right) = 0$, then by Lemma 5, γ lies on the line

$$\left\{ w_0 + \frac{t}{\sqrt{U(w_0; f)}} : t \in (-\infty, \infty) \right\},$$

which has the same direction as the vector field $\text{grad}\left[\, \Re \int \sqrt{U(w; f)}\, dw \,\right]$ at w_0. If $L_f\left(\frac{1}{f - w_0}\right) \neq 0$, then by Lemma 2

$$\Re \left\{ \frac{1}{U(w_0; f)} L_f\left(\frac{1}{f - w_0}\right) \right\} \geq 0. \tag{20}$$

Note that $U(w_0; f) \neq 0$, by Lemma 1(b), since $U(w; f) \not\equiv 0$ on γ. Also, by Lemma 5,

$$L_f\left(\frac{1}{f - w_0}\right) (dw)^2 > 0 \tag{21}$$

at w_0. Dividing (20) by (21), we find that

$$\Re \frac{1}{U(w_0; f)\,(dw)^2} \geq 0$$

at w_0. Therefore $\left|\arg[\sqrt{U(w_0; f)}\, dw]^2\right| \leq \pi/2$. We choose first a branch of $\sqrt{U(w; f)}$ on γ and then the tangent direction so that

$$\left|\arg[\sqrt{U(w_0 : f)}\, dw]\right| \leq \pi/4$$

at w_0. The conclusion then follows immediately.

Inserting into Theorem 2 conditions from Lemmas 1(d) and 4 that guarantee $U(w; f) \not\equiv 0$ and $L_f\left(\frac{1}{f - w}\right) \not\equiv 0$, we have the following consequence.

THEOREM 3. *Suppose* $\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q)$ *is compact,* $f \in \mathscr{F}$, $L \in H'(D)$ *is linearly independent of* ℓ_1 *and* ℓ_2, *and* $\Re L(f) = \max_{\mathscr{F}} \Re L$. *Assume furthermore that the supports of some representing measures for* ℓ_0 *and* L_f *do not separate the components of* $\overline{\mathbf{C}} - D$. *Then each nondegenerate component of* $\mathbf{C} - f(D)$ *is a single analytic arc whose tangent makes an angle at most* $\pi/4$ *with the vector field* $\text{grad}\left[\, \Re \int \sqrt{U(w; f)}\, dw \,\right]$.

The following is a notable special case:

THEOREM 3'. *Suppose* $D \neq \mathbf{C}$ *is simply connected,* $\mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2 P, Q)$ *is compact,* $f \in \mathscr{F}$, $L \in H'(D)$ *is linearly independent of* ℓ_1 *and* ℓ_2, *and* $\Re L(f) = \max_{\mathscr{F}} \Re L$. *Then each component of* $\mathbf{C} - f(D)$ *is an analytic arc whose tangent makes an angle at most* $\pi/4$ *with the vector field* $\text{grad}\left[\, \Re \int \sqrt{U(w; f)}\, dw \,\right]$.

We now apply Theorem 3 to the important families $\mathscr{S}(D, z_0)$ and $\mathscr{T}(D, p, q, P, Q)$:

THEOREM 4. *Suppose* $f \in \mathscr{S}(D, z_0)$, $L \in H'(D)$ *is not of the form* $L(g) = \alpha g(z_0) + \beta g'(z_0)$ (*for example, suppose* $L \not\equiv$ constant *on* $\mathscr{S}(D, z_0)$),

$\Re L(f) = \max_{\mathscr{F}} \Re L$, *and the support of some representing measure for* L *does not separate the components of* $\bar{\mathbf{C}}$ - D. *Then each nondegenerate component of* **C** - f(D) *is a single analytic arc whose tangent makes an angle at most* $\pi/4$ *with the radial direction. At most one component is unbounded.*

THEOREM 5. *Suppose* $f \in \mathscr{T}(D, p, q, P, Q)$, $L \in H'(D)$ *is not of the form* $L(g) = \alpha g(p) + \beta g(q)$ (*for example, suppose* $L \not\equiv$ constant *on* $\mathscr{T}$), $\Re L(f) = \max_{\mathscr{T}} \Re L$, *and the support of some representing measure for* L *does not separate the components of* $\bar{\mathbf{C}}$ - D. *Then each nondegenerate component of* **C** - f(D) *is an analytic arc whose tangent makes an angle at most* $\pi/4$ *with respect to the family of hyperbolae with foci* P *and* Q. *At most one component is unbounded.*

Proof of Theorems 4 *and* 5. The assertions are special cases of Theorem 3 for the examples (i) and (ii) (see (12) and (13)) except for the statement that branching does not occur at ∞. To see the latter, we use an idea of L. Brickman and D. Wilken [2]. If indeed two components of **C** - f(D) were unbounded, they would belong to a single component of $\bar{\mathbf{C}}$ - f(D) with at least two distinct points on all sufficiently large circles about the origin and on all sufficiently large ellipses with foci P and Q. In this situation, we have constructed a decomposition $f = \lambda f_1 + (1 - \lambda) f_2$, where $\lambda \in (0, 1)$ and f_1 and f_2 are in the family, but are not slit mappings (see [5, Theorems 1 and 2]). Since this is a convex decomposition, both f_1 and f_2 also maximize $\Re L$ over the family. This contradicts the assertion that extremal functions must be (analytic) slit mappings.

For simply connected domains, we may phrase Theorems 4 and 5 in terms of support points:

THEOREM 4'. *Suppose* $D \neq \mathbf{C}$ *is simply connected and* f *is a support point of* $\mathscr{S}(D, z_0)$. *Then* **C** - f(D) *consists of a single analytic arc whose tangent makes an angle at most* $\pi/4$ *with the radial direction.*

THEOREM 5'. *Suppose* $D \neq \mathbf{C}$ *is simply connected and* f *is a support point of* $\mathscr{T}(D, p, q, P, Q)$. *Then* **C** - f(D) *consists of a single analytic arc whose tangent makes an angle at most* $\pi/4$ *with respect to the family of hyperbolae with foci* P *and* Q.

Remarks. A special case of Theorem 4' is contained in the book by G. Goluzin [4, p. 147]. A Pfluger [7] and L. Brickman and D. Wilken [2] proved Theorem 4' in its general form. Theorem 5' appears to be a geometrically pleasing extension of their result, because as P, Q $\to$ 0, the hyperbolae degenerate into rays.

L. Brickman [1] and the present authors [5] have considered the extreme points, in the sense of convexity, of the families $\mathscr{S}(D, z_0)$ and $\mathscr{T}(D, p, q, P, Q)$. By definition, f is an *extreme point* of $\mathscr{F}$ if f belongs to $\mathscr{F}$ and there do not exist distinct $f_1, f_2 \in \mathscr{F}$ and $\lambda \in (0, 1)$ such that $f = \lambda f_1 + (1 - \lambda) f_2$. It was shown in [5] that the extreme points of $\mathscr{S}(D, z_0)$ and $\mathscr{T}(D, p, q, P, Q)$ have the weaker, but similar, properties that the boundary components are slits monotone relative to the family of circles about the origin and family of ellipses with foci P and Q, respectively.

Added in proof. In Theorems 4 and 5 it is proved that at most one component of **C** - f(D) is unbounded. The authors have recently shown that this conclusion holds, more generally, under the hypothesis of Theorem 3. The proof will appear shortly in C. R. Acad. Sci. Paris Sér. A.

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