AN APPLICATION OF UNIVERSAL ALGEBRA IN GROUP THEORY

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Our purpose in this note is to give two proofs of the following result.

THEOREM. *Suppose that* G *is a group and* H *a subgroup which is the intersection of subgroups of finite index in* G. *Then* H *is not conjugate in* G *to a proper subgroup of itself.*

The first proof, the line of reasoning which first led to the discovery of the theorem, is a very natural application of standard facts of universal algebra. Since it has other applications, one of which we shall mention at the end of this note, we feel that the idea is worth recording even though the second proof is a direct group-theoretic argument which shows a little more.

First Proof. We begin by defining some concepts of universal algebra which are perhaps better known in the narrower context of group theory. A congruence ρ on an algebra A is said to have *finite index* if the quotient algebra A/ρ is finite; an algebra A is said to be *residually finite* if, for each pair of distinct elements $a, b \in A$, there is a congruence ρ of finite index such that $a \not\equiv b \pmod{\rho}$; and an algebra A is said to be *Hopfian* if every surjective endomorphism of A is an automorphism. Our proof is based on the following generalization of a group-theoretic theorem often attributed to A. I. Mal'cev (see [4, pages 116 and 415]).

LEMMA. *Finitely generated residually finite algebras are Hopfian.*

This lemma has been known for many years, but since proofs do not appear in the texts, we include a proof which is a little simpler than that of T. Evans [1].

Proof of the lemma. Suppose that A is a finitely generated, residually finite algebra, that ϕ is a surjective endomorphism of A, and that a and b are distinct elements of A; we must prove that $a\phi \neq b\phi$. By hypothesis, there is a congruence ρ of finite index on A such that $a \not\equiv b \pmod{\rho}$. Define another congruence by

$$\sigma := \{(x, y) \in A \times A \mid x\psi = y\psi \text{ for all homomorphisms } \psi\colon A \to A/\rho\}\,.$$

Since A/ρ is finite and A is finitely generated, there are only finitely many homomorphisms $\psi\colon A \to A/\rho$, and so σ, being the intersection of finitely many congruences of finite index, is itself a congruence of finite index on A. Furthermore, if ψ is such a homomorphism, then so is $\phi\psi$. Thus, if two elements of A are congruent modulo σ, then so are their ϕ-images. It follows that ϕ induces a homomorphism $\bar{\phi}\colon A/\sigma \to A/\sigma$ which is surjective since ϕ is. Because A/σ is finite, $\bar{\phi}$ must also be injective. This means that, if two elements of A are not congruent modulo σ, then neither are their ϕ-images and, *a fortiori*, these images are distinct. But a and b are clearly not congruent modulo σ, for they have different images under the canonical projection from A to A/ρ. Therefore, $a\phi \neq b\phi$. (This proof showed, in effect, that every congruence of finite index on a finitely generated algebra contains a fully invariant congruence of finite index.)

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Now the first proof of the theorem is to be completed as follows. Suppose that H is an intersection of subgroups of finite index in the group G and that $g^{-1}Hg \leq H$. Let A be the G-space (G : H) consisting of all right cosets Hx of H in G, considered as an algebra with the elements of G acting on the right as unary operators. Since A is a transitive G-space, it is finitely generated (in fact, generated by any one of its elements); since congruences on A are in one-to-one correspondence with subgroups of G containing H, the hypothesis of our theorem is equivalent to the statement that A is residually finite. Therefore, by the lemma, A is Hopfian.

Put $K := gHg^{-1}$. Because $K \geq H$, there is a natural surjective homomorphism $\phi_1: (G:H) \to (G:K)$; because K and H are conjugate, there is also an isomorphism $\phi_2: (G:K) \to (G:H)$. The composite of these is a surjective endomorphism ϕ of A (described by $\phi: Hx \mapsto Hg^{-1}x$). Since A is Hopfian, ϕ is an isomorphism; hence ϕ_1 is injective, and this means that K = H. Therefore, $g^{-1}Hg = H$, as required.

Second Proof. As promised, we shall prove a slightly stronger theorem:

If H *is the intersection of subgroups whose normalisers have finite index in* G, *then* H *is not conjugate in* G *to a proper subgroup of itself.*

Let $\mathscr{X} := \{X \mid H \leq X \leq G \text{ and } |G : N(X)| \text{ is finite}\}$. If $X \in \mathscr{X}$, we put $\mathscr{C}(X) := \{x^{-1}Xx \mid x \in G,\ H \leq x^{-1}Xx\}$, which is a finite set since X has only finitely many conjugates in G.

Suppose now that $g^{-1}Hg \leq H$. If $Y \in \mathscr{C}(X)$ then $H \leq Y$, so $H \leq gHg^{-1} \leq gYg^{-1}$, and therefore also $gYg^{-1} \in \mathscr{C}(X)$. Thus, conjugation $Y \mapsto gYg^{-1}$ maps $\mathscr{C}(X)$ to itself. Since $\mathscr{C}(X)$ is finite and this map is clearly injective, it also is surjective. Now $\mathscr{X} = \bigcup_{X \in \mathscr{X}} \mathscr{C}(X)$, and therefore the map $X \mapsto gXg^{-1}$ is a permutation of $\mathscr{X}$. Hence,

$$H = \bigcap_{\mathscr{X}} X = \bigcap_{\mathscr{X}} (gXg^{-1}) = g\left(\bigcap_{\mathscr{X}} X\right)g^{-1} = gHg^{-1}\,.$$

Therefore, $g^{-1}Hg = H$, as required.

We mentioned in the first paragraph of this note that there are other applications of the idea behind our first proof. Here is another example. It has been well known for some time that in a finitely presented residually finite group the word problem is soluble (see C. F. Miller, [6, page 5]). Of course this has an analogue for other types of algebraic system.

LEMMA. *Let* $\mathfrak{V}$ *be a finitely defined variety of algebraic systems, and* A *a finitely presented algebra in* $\mathfrak{V}$. *If* A *is residually finite then the word problem in* A *is soluble.*

By applying this to the coset space of H in G as an algebra in the variety of G-spaces, we get the following result, implicit in [6, page 6].

THEOREM. *If* G *is a finitely presented group and* H *a finitely generated subgroup which is an intersection of subgroups of finite index, then the generalised word problem for* H *in* G *is soluble.*

This means that there is an algorithm to decide of any word in the generators of G whether or not the element it represents lies in H.

It should be observed that the restrictions imposed on $\mathfrak{V}$ and A in our statement of the lemma cannot easily be relaxed. For it is known that there exists a finitely generated, recursively presented, residually finite group G_0 whose word problem is not soluble (S. Meskin [5]). Taking $\mathfrak{V}$ to be the variety of groups and A to be G_0, we see that the lemma does not hold generally for recursively presented algebras in a finitely defined variety; and taking $\mathfrak{V}$ to be the variety of G_0-spaces and A to be its free algebra of rank 1 (which is the regular representation space for G), we see that the lemma does not hold generally for finitely presented algebras in a recursively definable variety.

The referee has kindly drawn our attention to references [2] and [3], in which our lemmas are to be found.

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