THE ABSOLUTE CONVERGENCE OF CERTAIN LACUNARY FOURIER SERIES

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Let G be a compact abelian group, and let Γ be its dual group. Suppose $E \subset \Gamma$ and f is a function on G. The function f is called an *E-function* if $\hat{f}(\gamma) = 0$ for all $\gamma \notin E$ ($\hat{f}$ is the Fourier transform of f). By A(G) we denote the space of functions whose transforms belong to $\ell^1(\Gamma)$, and $\|f\|_{A(G)}$ is defined to be $\|\hat{f}\|_{\ell^1(\Gamma)}$. For each set S(G) of functions defined on G, we denote by $S_E(G)$ the E-functions in S(G). A set $E \subset \Gamma$ is a *Sidon set* if $A_E(G) = C_E(G)$, where C(G) is the space of continuous functions on G. For $2 < p < \infty$, a set $E \subset \Gamma$ is a $\Lambda(p)$-*set* if $L^2_E(G) = L^p_E(G)$. A set $E \subset \Gamma$ is a Λ-*set* if it is a $\Lambda(p)$-set for all p and if in addition the inclusions $L^2_E(G) \to L^p_E(G)$ have norm at most $Cp^{1/2}$, where C depends only on the set E.

It is known that every Sidon set is a Λ-set [9, p. 128], and that there exist sets that are $\Lambda(p)$-sets for all p but are not Sidon sets [2, p. 803]. Actually, in the light of results in [1, p. 131], the sets constructed in [2] are not Λ-sets. It is therefore natural to ask whether there exist Λ-sets that are not Sidon sets. In general, this is an open question, but in certain torsion groups every Λ-set is also a Sidon set [6]. That Sidon sets are close to Λ-sets from a structural standpoint was shown in [1]. In this paper, we show that in an analytical sense they are also close. In particular, we construct a Banach space B(G) of functions on G such that $A(G) \hookrightarrow B(G) \hookrightarrow C(G)$ and such that $E \subset \Gamma$ is a Λ-set if and only if $A_E(G) = B_E(G)$. The construction of B(G) is motivated by the work in [3] and [4], and the connection between B(G) and Λ-sets is analogous to the connection between A. Figà-Talamanca's $A^p(G)$-spaces and $\Lambda(p)$-sets [5].

In Section 1 of this paper, we define two spaces K(G) and R(G) of functions on G that, (in the language of M. A. Rieffel [7]) are Banach modules. The space B(G) is then defined, and it turns out to be a realization of the Banach module tensor product $K(G) \otimes_{L^1(G)} R(G)$. In Section 2, we establish the connection between B(G) and Λ-sets.

1. DEFINITIONS AND PROPERTIES OF THE BASIC SPACES

For $f \in \bigcap_{2<p<\infty} L^p(G)$, let

$$\|f\|_\Lambda = \sup\{p^{-1/2}\|f\|_p \mid 2 < p < \infty\},$$

and let K(G) be the set of all functions f on G for which $\|f\|_\Lambda$ is finite. It is easy to verify that $\|\ \|_\Lambda$ is a norm on K(G) and that, endowed with this norm, K(G) becomes a two-sided Banach $L^1(G)$-module with respect to convolution. Next, we shall define a space R(G) of functions that is also a two-sided Banach $L^1(G)$-module

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and whose dual is K(G). This is done as follows. Let R(G) be the closure in K(G)* of the set of continuous functions on G that define in the natural way elements of K(G)*. We shall see that every continuous function defines an element of K(G)*. In the following proposition, we collect some relevant facts about K(G) and R(G).

PROPOSITION 1. *Let* K(G) *and* R(G) *be defined as above, and denote by* $\|\ \|_R$ *the norm dual to* $\|\ \|_\Lambda$. *The following hold.*

(1) $L^1(G) \supset R(G) \supset L^q(G)$ *for all* q $(1 < q \leq 2)$, *and*

$$2^{1/2}\|\ \|_1 \leq \|\ \|_R \leq (q/(q-1))^{1/2}\|\ \|_q .$$

(2) $L^p(G) \supset K(G) \supset L^\infty(G)$ *for all* p $(2 \leq p < \infty)$, *and*

$$p^{-1/2}\|\ \|_p \leq \|\ \|_\Lambda \leq 2^{-1/2}\|\ \|_\infty .$$

(3) $(R(G)^*, \|\ \|_{R^*}) = (K(G), \|\ \|_\Lambda)$.

(4) *If* $f \in K(G)$ *and* $g \in R(G)$, *then* $f*g \in C(G)$ *and* $\|f*g\|_\infty \leq \|f\|_\Lambda \|g\|_R$.

Proof. (2) follows immediately from the definition of K(G). To see that (1) holds, take $g \in L^p(G)$ and $f \in K(G)$. Then, for $p = q/(q-1)$,

$$\left|\int fg\right| \leq \|g\|_q \|f\|_p \leq (p^{1/2}\|g\|_q)\|f\|_\Lambda .$$

This shows that $\|g\|_R \leq (q/(q-1))^{1/2}\|g\|_q$ for all $g \in L^q(G)$. Next, consider the inclusion map i: $C(G) \rightarrow K(G)$ and its adjoint i^*: $K(G)^* \rightarrow M(G)$. We have the relations

$$\|i^*\| = \|i\| = \sup\{\|f\|_\Lambda/\|f\|_\infty \mid f \in C(G)\},$$

and by (2) the last member is at most $2^{-1/2}$. Taking $g \in C(G)$ and noting that $i^*(g) = g$, we see that $\|g\|_1 \leq 2^{-1/2}\|g\|_R$. Once we have shown that $L^1(G) \supset R(G)$, this norm inequality will hold for all $g \in R(G)$. We now show that $L^1(G) \supset R(G)$. Suppose $\{g_n\}$ is a Cauchy sequence in $\|\ \|_R$ of continuous functions. Since $\|\ \|_R$ is stronger than $\|\ \|_1$, the sequence $\{g_n\}$ converges to some g in $L^1(G)$, and some subsequence $\{g_{n_k}\}$ converges pointwise almost everywhere to g. Given $\varepsilon > 0$, take N so that $n, m > N$ implies $\|g_n - g_m\|_R < \varepsilon$. Consider any $f \in K(G)$. Clearly,

$$\left|\int (g - g_n)f\right|/\|f\|_\Lambda \leq \int \lim_{k\to\infty} |g_{n_k} - g_n|\,|f|/\|f\|_\Lambda$$

$$\leq \liminf_{k\to\infty} \int |g_{n_k} - g_n|\,|f|/\|f\|_\Lambda \leq \liminf \|g_{n_k} - g_n\|_R \leq \varepsilon .$$

Therefore $g - g_n \in R(G)$ and $\|g - g_n\|_R \leq \varepsilon$. It follows that $g \in R(G)$ and $L^1(G) \supset R(G)$.

Next, we show that $R(G)^* = K(G)$. Since the inclusion map i: $L^q(G) \rightarrow R(G)$ is dense for all $q > 1$, the adjoint i^*: $R(G)^* \rightarrow L^p(G)$ is one-to-one for all $p < \infty$. Therefore $R(G)^* \subset \bigcap_{p<\infty} L^p(G)$. Take any $f \in R(G)^*$ and any $p < \infty$. Then there exists a $g \in L^q(G)$ such that $\left|\int fg\right| = \|f\|_p \|g\|_q$. Hence, by (1),

$$\|f\|_{R^*} \geq \left|\int fg\right| / \|g\|_R \geq \|f\|_p \|g\|_q / p^{1/2} \|g\|_q .$$

Therefore $\|f\|_{R^*} \geq \|f\|_\Lambda$, and $R(G)^* \subset K(G)$. On the other hand, for $f \in K(G)$ and $g \in R(G)$, we have the inequality $\left|\int fg\right| \leq \|f\|_\Lambda \|g\|_R$. Therefore $\|f\|_\Lambda \geq \|f\|_{R^*}$ and $K(G) \subset R(G)^*$.

Finally, we show that (4) holds. For each $g \in R(G)$, the translation map $\tau_g: G \to R(G)$ defined by $(\tau_g(x))(y) = g(y - x)$ is continuous. This follows from the facts that $\tau_f: G \to C(G)$ is continuous, that $C(G)$ is dense in $R(G)$, and that $\| \ \|_\infty$ is a stronger norm than $\| \ \|_R$. Now suppose $f \in K(G)$ and $g \in R(G)$. Then (4) follows immediately from the translation invariance of the $\| \ \|_\Lambda$- and $\| \ \|_R$-norms and the continuity of $\tau_g: G \to R(G)$.

We define $B(G)$ to be the set of functions f on g with a representation

$$f(x) = \sum_{n=1}^{\infty} \tilde{f}_n * g_n(x),$$

where $\tilde{f}_n(x) = f_n(-x)$, $f_n \in K(G)$, $g_n \in R(G)$, and $\sum_{n=1}^{\infty} \|f_n\|_\Lambda \|g_n\|_R < \infty$. In view of (4) of Proposition 1, the series for f converges uniformly. Therefore $B(G)$ is a set of continuous functions. We give $B(G)$ a norm by putting

$$\|f\|_B = \inf \sum_{n=1}^{\infty} \|f_n\|_\Lambda \|g_n\|_R,$$

where the infimum is taken over all representations of f. The features of $B(G)$ that will concern us follow from some general theorems about tensor products of Banach modules (see [7] and [8]), once we have observed that $B(G)$ is in fact such a tensor product. It is easily seen that $K(G)$ is a (two-sided) Banach $L^1(G)$-module with respect to convolution. That is, for $\phi \in L^1(G)$ and $f \in K(G)$, the action of ϕ on f is given by $\phi * f$. We see that $R(G)$ is a (two-sided) Banach $L^1(G)$-module, by letting the action of $\phi \in L^1(G)$ on $g \in R(G)$ be given by $\tilde{\phi} * g$. An almost word-for-word translation of [8, p. 76, Theorem 3.3] in terms of $K(G)$ and $R(G)$ shows that $K(G) \otimes_{L^1(G)} R(G)$ is isometrically isomorphic to $B(G)$, the mapping being such that $f \otimes g$ corresponds to $\tilde{f} * g$. Thus $B(G)$ is a Banach space, and

$$\mathrm{Hom}_{L^1(G)}(K(G), R(G)^*) \cong (K(G) \otimes_{L^1(G)} R(G))^* .$$

The importance of this result lies in the fact that $\mathrm{Hom}_{L^1(G)}(K(G), R(G)^*)$ is the space of multipliers from $K(G)$ to $R(G)^*$, and that $R(G)^* = K(G)$. Thus, denoting by $M(K)$ the multipliers from $K(G)$ to $K(G)$, we see that

$$M(K) \cong B(G)^* .$$

Moreover, this isomorphism is such that for each $\nu \in B(G)^*$ and its corresponding element $T_\nu \in M(K)$,

$$\langle g, T_\nu f \rangle = \langle \tilde{f} * g, \nu \rangle,$$

where $\langle\ ,\ \rangle$ on the left denotes the dual pairing between $R(G)$ and $R(G)^* = K(G)$, and $\langle\ ,\ \rangle$ on the right denotes the dual pairing between $B(G)$ and $B(G)^*$. Another way of expressing this correspondence is to say that, for $\nu \in B(G)^*$, the function on Γ associated with $T_\nu \in M(K)$ is $\hat{\nu}$.

2. THE CONNECTION WITH Λ-SETS

If $E \subset \Gamma$, then $K_E(G)$, $R_E(G)$, and $B_E(G)$ are all closed subspaces of $K(G)$, $R(G)$, and $B(G)$, respectively. This is an immediate consequence of the fact that the norms on these spaces are stronger than the $L^1(G)$-norm.

PROPOSITION 2. *If* $E \subset \Gamma$ *is a* Λ*-set, then*

$$K_{-E}(G) \otimes_{L^1(G)} R_E(G) \cong B_E(G),$$

and the $\| \ \|_B$*-norm is equivalent to the* $\| \ \|_{K_{-E} \otimes R_E}$*-norm on* $B_E(G)$, *although not isometric to it.*

Proof. Clearly, $K_{-E}(G) \otimes_{L^1(G)} R_E(G) \subset B_E(G)$ and $\| \ \|_{K_{-E} \otimes R_E} \geq \| \ \|_B$.

To prove the other inclusion, consider χ_E, the characteristic function of E, which defines a multiplier T from $L^2(G)$ to $L^2_E(G)$ by the relation $\widehat{Tf} = \chi_E \hat{f}$. Noting that E is a Λ-set if and only if $L^2_E(G) = K_E(G)$, we see, by restricting T to $K(G)$, that $T \in M(K)$ and that the range of T is $K_E(G)$. To estimate the norm of T, take $f \in K(G)$, and suppose that C is the Λ-constant of E. Then

$$\|Tf\|_\Lambda \leq C\,\|Tf\|_2 \leq C\,\|f\|_2 \leq C\,2^{1/2}\,\|f\|_\Lambda,$$

and therefore $\|T\|_{M(K)} \leq 2^{1/2} C$.

Suppose $f \in B_E(G)$, where $f = \sum_{n=1}^\infty \tilde{f}_n * g_n$ is a typical representation. Then

$$f = T(f) = T\left(\sum_{n=1}^{\infty} \tilde{f}_n * g_n\right) = \sum_{n=1}^{\infty} T(\tilde{f}_n * g_n) = \sum_{n=1}^{\infty} (T\tilde{f}_n) * g_n .$$

Note that $T\tilde{f}_n = \tilde{h}_n$, where $h_n \in K_{-E}(G)$. In fact, $\hat{h}_n = \chi_{-E} \hat{f}_n$. Moreover, $\|h_n\|_\Lambda \leq 2^{1/2} C \|f_n\|_\Lambda$.

Since $S \in M(K)$, it follows that the adjoint S^* belongs to $M(K^*)$. The restriction of S^* to $R(G)$ yields a multiplier from $R(G)$ to $R_E(G)$. This follows easily from the facts that $R(G)$ is closed in $K(G)^*$ and that the continuous functions are dense in $R(G)$ and are contained in $K(G)$. Therefore

$$f = T^*(f) = \sum_{n=1}^{\infty} \tilde{h}_n * (T^* g_n),$$

where $\|T^* g_n\|_R \leq C\, 2^{1/2} \|g_n\|_R$. Hence $f \in K_{-E}(G) \otimes_{L^1(G)} R_E(G)$ and $\|f\|_{K_{-E} \otimes R_E} \leq 2\, C^2 \|f\|_B$.

We remark that for each $E \subset \Gamma$,

$$A_E(G) \cong L^2_{-E}(G) \otimes_{L^1(G)} L^2_E(G) \quad \text{and} \quad \|\ \|_A = \|\ \|_{L^2_{-E} \otimes L^2_E}.$$

In preparation for the main theorem, we shall prove a probabilistic result, for which we need the following notation. Let Ω be the interval $[0, 1]$, and let p denote Lebesgue measure. Consider the Rademacher functions $\varepsilon_n\colon \Omega \to \{-1, 1\}$ defined by $\varepsilon_n(\omega) = (-1)^{m-1}$ if $\omega \in [(m-1)2^{-n}, m\, 2^{-n})$ and $\varepsilon_n(1) = -1$, where $n = 0, 1, 2, \cdots$. For each $f \in L^2(G)$, select an enumeration $\gamma_0, \gamma_1, \cdots$ of the support of $\hat{f}$, and define $f_\omega \in L^2(G)$ by $\hat{f}_\omega(\gamma_n) = \varepsilon_n(\omega) \hat{f}(\gamma_n)$.

PROPOSITION 3. *If* $f \in L^2(G)$, *then the following statements hold.*

(a) $p\{\omega \in \Omega \mid f_\omega \in K(G)\} = 1$.

(b) *For each* $\varepsilon > 0$, *there exists an* N (*depending on* f *and* ε) *such that* $\|f_\omega\|_\Lambda \leq N \|f\|_2$ *for all* ω *except possibly a set of measure less than* ε.

(c) *For each* $f \in L^2(G)$, *there exists an* $\omega \in \Omega$ *such that* $\|f_\omega\|_\Lambda \leq 16 \|f\|_2$.

Proof. We shall use the following well-known theorem [10, p. 214]. *Let* $\{u_n\}$ *be a sequence of complex numbers such that* $\sum_{n=0}^{\infty} |u_n|^2 = r^2 < \infty$, *and let* $g(\omega) = \sum_{n=0}^{\infty} u_n \varepsilon_n(\omega)$. *Then* $\exp(\lambda |g(\cdot)|^2) \in L^1(\Omega)$ *for every* $\lambda > 0$, *and*

$$\int_\Omega \exp(\lambda |g(\omega)|^2)\, d\omega = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_\Omega |g(\omega)|^{2k}\, d\omega \leq \sum_{k=0}^{\infty} \frac{k^k}{k!} (4\lambda r^2)^k \leq \sum_{k=0}^{\infty} (4e\lambda r^2)^k .$$

Consider the series

$$\sum_{n=0}^{\infty} \hat{f}(\gamma_n) \langle x, \gamma_n \rangle \varepsilon_n(\omega) . \tag{1}$$

Since $f \in L^2(G)$, we can apply the theorem to the series (1) with $e\lambda = (8r^2)^{-1}$, and we obtain the inequality

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_\Omega |f_\omega(x)|^{2k}\, d\omega \leq 2 .$$

Integrating both sides over G and switching sums and integrals, we see that

$$\int_\Omega \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \|f_\omega\|_{2k}^{2k} \right) d\omega \leq 2 . \tag{2}$$

Put $E_n = \{\omega \mid$ integrand in (2) is at most $2n\}$; then clearly $p(\Omega \setminus E_n) \leq 1/n$ and $p(E_n) \geq 1 - 1/n$. If $\omega \in E_n$, then

$$\frac{\lambda^k}{k!} \|f_\omega\|_{2k}^{2k} \leq 2n \qquad \text{for } k = 1, 2, \cdots .$$

Using $e\lambda = (8r^2)^{-1}$ and $k! \leq (k+1)^{k+1}/e^k$, we see that

(3) $$(2k)^{-1/2} \|f_\omega\|_{2k} \leq ((k+1)/k)^{1/2} (k+1)^{1/2k} (2n)^{1/2k}\, 2r .$$

Hence there exists a constant K_n such that

$$(2k)^{-1/2} \|f_\omega\|_{2k} \leq K_n \|f\|_2 \qquad \text{for all } k = 1, 2, \cdots .$$

Therefore $\|f_\omega\|_\Lambda \leq 2^{1/2} K_n \|f\|_2$ for all $\omega \in E_n$; this proves (a) and (b). As for (c), set $n = 2$ in (3).

PROPOSITION 4. $A(G) \subset B(G) \subset C(G)$ *and* $2^{9/2} \| \ \|_A \geq \| \ \|_B \geq \| \ \|_\infty$.

Proof. Take $f \in B(G)$, where $f = \sum_{n=1}^\infty \tilde{f}_n * g_n$. Then

$$\|f\|_\infty \leq \sum_{n=1}^\infty \|\tilde{f}_n * g_n\|_\infty \leq \sum_{n=1}^\infty \|f_n\|_\Lambda \|g_n\|_R .$$

Since this holds for all representations, $\|f\|_\infty \leq \|f\|_B$. That $B(G) \subset C(G)$ was observed earlier.

Now take $h \in A(G)$. Choose f and $g \in L^2(G)$ so that $\hat{\tilde{f}}\hat{g} = \hat{h}$ and $|\hat{\tilde{f}}| = |\hat{g}| = |\hat{h}|^{1/2}$. By Proposition 3, there exists a sign change $\omega \in \Omega$ such that $\tilde{f}_\omega \in K(G)$ and $\|\tilde{f}_\omega\|_\Lambda \leq 16 \|\tilde{f}\|_2$. Note that $\widehat{\tilde{f}_\omega}\hat{g}_\omega = \hat{\tilde{f}}\hat{g}$ and hence $\tilde{f}_\omega * g_\omega = h$. Thus $\tilde{f}_\omega * g_\omega$ is a representation of h as an element of $B(G)$, and

$$\|h\|_B \leq \|\tilde{f}_\omega\|_\Lambda \|g_\omega\|_R \leq 16 \|\tilde{f}\|_2 2^{1/2} \|g_\omega\|_2 = 2^{9/2} \|\hat{\tilde{f}}\|_2 \|\hat{g}\|_2 = 2^{9/2} \|\hat{h}\|_1 .$$

Finally we prove the main result of this paper.

THEOREM. $E \subset \Gamma$ *is a* Λ-*set if and only if* $A_E(G) = B_E(G)$.

Proof. We observe first that $(L^2_E(G))^* = L^2_E(G)$. Now suppose E is a Λ-set. Then, as we noted earlier, $L^2_E(G) = K_E(G)$. Take $g \in R_E(G)$. Then

$$\|g\|_R \geq \sup_{f \in K_E(G)} \frac{|\langle f, g\rangle|}{\|f\|_\Lambda} \geq \sup_{f \in K_E(G)} \frac{|\langle f, g\rangle|}{C\|f\|_2}$$

$$= \frac{1}{C} \sup_{f \in L^2_E(G)} \frac{|\langle f, g\rangle|}{\|f\|_2} = \frac{1}{C} \|g\|_{(L^2_E(G))^*} = \frac{1}{C} \|g\|_2 .$$

Hence $R_E(G) = L^2_E(G)$, $K_{-E}(G) = L^2_{-E}(G)$, and therefore

$$K_{-E}(G) \otimes_{L^1(G)} R_E(G) = L^2_{-E}(G) \otimes_{L^1(G)} L^2_E(G) .$$

Using Proposition 2 and the fact that $L^2_{-E}(G) \otimes_{L^1(G)} L^2_E(G)$ is isometrically equal to $A_E(G)$, we see that $A_E(G) = B_E(G)$.

Now suppose $A_E(G) = B_E(G)$. Then $B_E(G)^* = A_E(G)^*$. Since $A_E(G)^*$ can be identified with $\ell^\infty(E)$, the relation $B_E(G)^* = A_E(G)^*$ implies the existence of a constant C with the property that for each $\phi \in \ell^\infty(E)$ and each $\varepsilon > 0$, there exists a $\nu \in B(G)^*$ such that $\hat{\nu}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$ and such that $\|\nu\|_{B^*} \leq C\|\phi\|_\infty + \varepsilon$. Since $B(G)^*$ can be identified with M(K), there exists a $T \in M(K)$ such that $\widehat{Tf} = \hat{\nu}\hat{f}$ and $\|T_\nu\|_{M(K)} \leq C\|\phi\|_\infty + \varepsilon$.

Now suppose $f \in L^2_E(G)$. By Proposition 3, there exists a ± 1-valued function ϕ on E such that $\phi\hat{f}$ is the Fourier transform of a function in $K_E(G)$. Since $\hat{f}(\gamma) = 0$ for all $\gamma \notin E$, we see that

$$\hat{f} = \phi(\phi\hat{f}) = \hat{\nu}(\hat{\nu}\hat{f}) = \hat{\nu}(\widehat{Tf}) = (T(Tf))^\wedge,$$

and therefore $f = T(Tf)$. But $Tf \in K(G)$, by the choice of ϕ; moreover, $T \in M(K)$; therefore $f \in K_E(G)$ and $\|f\|_\Lambda \leq (C + \varepsilon)\, 16\, \|f\|_2$. This shows that $L^2_E(G) = K_E(G)$, and this is equivalent to the assertion that E is a Λ-set.

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