CONDITIONALLY CONVERGENT SERIES IN R^∞

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1. INTRODUCTION

Let A denote the infinite series $\sum_{k=1}^{\infty} a_k$, where $\{a_k\}_{k=1}^{\infty}$ is a sequence of elements of a topological vector space X. If p is a permutation of the positive integers, let A_p denote the series $\sum_{k=1}^{\infty} a_{p(k)}$, called a *rearrangement* of A. Let S_A denote the set of elements $s \in X$ such that some rearrangement of A converges to s. If A converges and S_A contains only one element, then A is said to *converge with invariant sum.* If A converges, but not every rearrangement of A converges, then A is said to *converge conditionally.* If A_p converges for every permutation p, then A is said to *converge unconditionally.*

In every linear topological space, unconditional convergence implies convergence with invariant sum. In a Euclidean space R^m, the converse is true. In fact, if A is a conditionally convergent series in R^m, then S_A is an affine subspace of R^m whose dimension is at least one. (In the case when m = 1, this result is of course a well-known theorem of Riemann (see [15, p. 419] or [1, Chapter 12]); proofs for the general case have been given by E. Steinitz [13] and others ([6], [14], [16], [17]).) In Section 2, we shall prove that the same statement holds for the countably-infinite product space R^∞ (with the product topology). Our treatment makes it easy to understand just how the dimension of S_A is determined, in either the finite- or infinite-dimensional case.

C. W. McArthur [11], using work of H. Hadwiger [9], showed that in every infinite-dimensional Banach space there is a conditionally convergent series that converges with invariant sum. His method yields the same result for every infinite-dimensional Fréchet space on which a continuous homogeneous norm can be defined. A Fréchet space has such a norm if and only if it does not contain a subspace isomorphic to R^∞ (see [2]).

We should like to mention the important result of A. Dvoretzky and C. A. Rogers [5], that in every infinite-dimensional Banach space there is a series that converges unconditionally but not absolutely. For other proofs of this, see [10], [12], and [7] or [8].

In Section 3, we consider another question about series in R^∞: Is it true that for every sequence $\{a_k\}_{k=1}^{\infty}$ in R^∞ such that $\lim_{k\to\infty} a_k = 0$, there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$, with each ε_k equal to +1 or -1, such that $\sum_{k=1}^{\infty} \varepsilon_k a_k$ converges? The answer is yes. The answer was known to be yes in the case of R^m [3] and no in the case of every infinite-dimensional Banach space [4, p. 157, Theorem 8].

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2. CONDITIONALLY CONVERGENT SERIES

In order to state our main theorem, we need a few more definitions. If W is a subspace of X, if $\pi = \pi_W$ is the projection from X onto W, and if A is a series $\sum_{k=1}^{\infty} a_k$ in X, let πA denote the series $\sum_{k=1}^{\infty} \pi a_k$ in W. It is easy to see that $\pi S_A \subseteq S_{\pi A}$, and that this inclusion is sometimes proper.

If A is a series in X, and B is a series in Y, and if $X \times Y$ is the product space, let $A \times B$ denote the series in $X \times Y$ such that $\pi_X(A \times B) = A$ and $\pi_Y(A \times B) = B$.

In R^m or R^∞, let $\{e_j\}$ denote the canonical basis, and let σ_j and τ_j denote projections, as follows:

$$\sigma_j \left(\sum_i x_i e_i \right) = x_j e_j,$$

$$\tau_j \left(\sum_i x_i e_i \right) = \sum_{i=1}^{j} x_i e_i .$$

Given two series A and B, we shall say that A *determines* B if for every permutation p such that A_p converges, B_p also converges.

In describing S_A for the case of an arbitrary conditionally convergent series A in R^∞, we may ignore the case when $\sigma_j A$ is unconditionally convergent for one or more values of j, since it is obvious that $\sigma_j S_A$ is a singleton for every such j.

THEOREM 1. *Let* A *be a series in* R^∞ *such that for every* j, *the series* $\sigma_j A$ *is conditionally convergent to a sum* x_j. *Let* J *be the set of indices* j *such that* $j \geq 2$ *and* $\tau_{j-1} A$ *determines* $\sigma_j A$. *Then there exist linear mappings* L_j *from* $\tau_{j-1} R^\infty$ *onto* R *such that*

$$S_A = \{ s = \{s_j\}_{j=1}^{\infty} \in R^\infty : s_j = x_j + L_j(s_1, \cdots, s_{j-1}) \text{ for each } j \in J \} .$$

The theorem is proved by means of the technical proposition below. For $x = (x_1, \cdots, x_m) \in R^m$, let $|x| = \left(\sum_{j=1}^{m} x_j^2 \right)^{1/2}$. We shall use the symbol α_p to mean the sum of a convergent rearrangement A_p of the series A, and similarly, β_r to mean the sum of B_r, and so forth.

PROPOSITION. *Let* A *and* B *be series in* R^m *and* R, *respectively, that converge conditionally to zero. If* A *determines* B, *then there is a linear mapping* L *from* R^m *onto* R *such that* $\sum_{k=1}^{\infty} |b_k - L(a_k)| < \infty$, *so that*

$$S_{A \times B} = \{ (x, y) \in R^m \times R : x \in S_A \text{ and } y = L(x) \} .$$

If A *does not determine* B, *then* $S_{A \times B} = S_A \times S_B$. *In this case, in fact, if* $\alpha_p \in S_A$, $\beta_r \in S_B$, *and* $\varepsilon > 0$, *then for all sufficiently large* k_0 *there is a permutation* q *such that* $\alpha_q = \alpha_p$, $\beta_q = \beta_r$, $q(k) = p(k)$ *for* $k \leq k_0$, *and* $\left| \sum_{k=k_0}^{r} a_{q(k)} \right| < \varepsilon$ *for all* $r \geq k_0$.

How the proposition implies the theorem. We may suppose without loss of generality that $x_j = 0$ for every j. Let X be the subspace of R^∞ spanned by the elements e_j for which $j \notin J$. Suppose that we can show that $S_{\pi_X A} = X$. Then, if J is void, $S_A = R^\infty$. Otherwise, according to the first part of the proposition, for each $j \in J$ there exists a linear map L_j from $\tau_{j-1} R^\infty$ onto $\sigma_j R^\infty$ such that $\sum_{k=1}^{\infty} |\sigma_j a_k - L_j(\tau_{j-1} a_k)| < \infty$, and therefore

$$S_A = \{s = \{s_j\}_{j=1}^{\infty} \in R^\infty : s_j = L_j(s_1, \cdots, s_{j-1}) \text{ for each } j \in J\}.$$

It remains to show that $S_{\pi_X A} = X$. It suffices to deal with the case when $X = R^\infty$, and to show that then $S_A = R^\infty$. Let $s = \{s_k\}_{k=1}^{\infty} \in R^\infty$. For each k, there exists a permutation p_k such that $(\sigma_k A)_{p_k}$ converges to s_k. We must show that there exists a permutation p such that A_p converges to s. We may suppose, without loss of generality, that $s = 0$.

We shall define a strictly increasing sequence of integers k(m) and a sequence of permutations q_m such that

(1) $(\tau_{m+1} A)_{q_m}$ converges to zero,

(2) $m \in \{q_m(j) : j \leq k(m)\}$,

(3) $q_m(j) = q_{m-1}(j)$ for $m \geq 1$ and $j \leq k(m)$, and

(4) $\left|\sum_{j=k(m)}^{r} \tau_m a_{q_m(j)}\right| < 2^{-m}$ for $m \geq 1$ and $r \geq k(m)$.

Then we shall let $p(j) = \lim_{m \to \infty} q_m(j)$. By (2) and (3), it is clear that p is a permutation. By (3) and (4), we see that

(5) $\left|\sum_{j=k(m)}^{r} \tau_m a_{p(j)}\right| < 2^{-m+1}$ for $m \geq 1$ and $r \geq k(m)$.

By (1), (4), and (5), A_p converges to zero.

It remains to specify the definition of k(m) and q_m. Let $q_0 = p_0$, and choose k(0) sufficiently large so that (2) is satisfied for $m = 0$. Now suppose that q_j and k(j) have been chosen suitably for $j < m$. Then the series $\tau_m A$ does not determine the series $\sigma_{m+1} A$, and both $(\tau_m A)_{q_{m-1}}$ and $(\sigma_{m+1} A)_{q_{m-1}}$ converge to zero. According to the proposition, then, we may choose k(m) sufficiently large and find a q_m such that (1) to (4) are satisfied. The argument is complete.

It remains to prove the proposition. The proof will unfold in a sequence of lemmas.

LEMMA 1 (see [6], [13]). *Suppose that* $b_j \in R^m$ *for* $1 \leq j \leq n$, *that* $\left|\sum_{j=1}^{n} b_j\right| < \delta$, *and that* $|b_j| < \delta$ *for each* j. *Then there is a permutation* p *of the integers from* 1 *to* n *such that* $\left|\sum_{j=1}^{k} b_{p(j)}\right| < (2^m - 1)\delta$ *for* $1 \leq k \leq n$.

The next lemma is an easy consequence of Lemma 1.

LEMMA 2. *Let* A *be a convergent series in* R^m. *Suppose that there is a permutation* q *such that a subsequence* $\left\{\sum_{j=i}^{n(i)} a_{q(j)}\right\}_{i=1}^{\infty}$ *of the partial sums of* A_q *converges to* s. *Then there is a permutation* p *such that* A_p *converges to* s.

LEMMA 3. *Let* A *be a series in* R^m *that converges to zero. Let* T_A *be the set of* $s \in R^m$ *such that for every* $\varepsilon > 0$ *and* N, *there is a finite set* X *of integers greater than* N *such that* $\left|s - \sum_{n \in X} a_n\right| < \varepsilon$. *Then* $S_A = T_A$.

Proof. Let $\alpha_p \in S_A$. Let $\varepsilon > 0$ and N be arbitrary. Choose $m > N$ so that $\left|\sum_{n=1}^{m} a_n\right| < \varepsilon/2$. Choose k so that $\left|\alpha_p - \sum_{j=1}^{k} a_{p(j)}\right| < \varepsilon/2$ and so that the set $Z = \{p(j): 1 \le j \le k\}$ contains the set $Y = \{n: 1 \le n \le m\}$. If $X = Z \setminus Y$, then $\left|\alpha_p - \sum_{n \in X} a_n\right| < \varepsilon$. Therefore $\alpha_p \in T_A$.

For $s \in T_A$, the following inductive procedure defines a permutation q such that a subsequence $\left\{\sum_{j=1}^{n(k)} a_{q(j)}\right\}_{k=1}^{\infty}$ of partial sums converges to s. It follows that $T_A \subset S_A$.

Step 1. Choose m(1) so that $\left|\sum_{n=1}^{m(1)} a_n\right| < 1/2$. Let $q(j) = j$ for $1 \le j \le m(1)$. Set k equal to 1 and proceed to Step 2.

Step 2. Let X be a finite set of integers such that

$$X \cap \{q(j): 1 \le j \le m(k)\} = \emptyset \quad \text{and} \quad \left|s - \sum_{n \in X} a_n\right| < 2^{-k}.$$

Pick n(k) and define q on $\{j: m(k) < j \le n(k)\}$ so that $\{q(j): m(k) < j \le n(k)\}$ is a one-to-one enumeration of X. Note that then $\left|s - \sum_{j=1}^{n(k)} a_{q(j)}\right| < 2^{1-k}$. Proceed to Step 3.

Step 3. Choose $m(k+1)$ so that $\left|\sum_{n=1}^{m(k+1)} a_n\right| < 2^{-k-1}$ and so that the set $Z = \{n: 1 \le n \le m(k+1)\}$ contains the set $Y = \{q(j): 1 \le j \le n(k)\}$. Define q on $\{j: n(k) < j \le m(k+1)\}$ so that $\{q(j): n(k) < j \le m(k+1)\}$ is a one-to-one enumeration of the integers in $Z \setminus Y$. Change the value of k by adding 1, and proceed to Step 2.

The inductive procedure is fully described. Lemma 3 is proved.

LEMMA 4. *If* A *is a series in* R^m *that converges to zero, then* S_A *is a linear subspace.*

Proof. It suffices to show (1) that if s_1 and s_2 belong to S_A, then so does $s_1 - s_2$; and (2) that if $s \in S_A$ and $0 < \lambda < 1$, then $\lambda s \in S_A$.

To prove (1), we shall show that $s_1 - s_2 \in T_A$. Let $\varepsilon > 0$ and N be arbitrary. There is a finite set Y containing every $n \le N$ such that $\left|s_2 - \sum_{n \in Y} a_n\right| < \varepsilon/2$. There is a finite set Z containing Y such that $\left|s_1 - \sum_{n \in Z} a_n\right| < \varepsilon/2$. Let $X = Z \setminus Y$. Then $\left|s_1 - s_2 - \sum_{n \in X} a_n\right| < \varepsilon$. Therefore $s_1 - s_2 \in T_A$. To prove (2) we shall show that $\lambda s \in T_A$. Let $\varepsilon > 0$ and N be arbitrary. We may suppose that N is sufficiently large so that $|a_n| < \varepsilon$ for $n > N$. Since $s \in T_A$, there is a finite set X of integers greater than N such that $\left|s - \sum_{n \in X} a_n\right| < \varepsilon$. There is an orthonormal basis $\{e_1, \cdots, e_m\}$ such that $s = |s|\, e_m$. Since $|a_n| < \varepsilon$ for $n \in X$ and $\left|\sum_{n \in X} \tau_{m-1} a_n\right| < \varepsilon$, Lemma 1 guarantees the existence of an enumeration $X = \{n(j)\}_{j=1}^{k}$ such that

$$\left|\sum_{j=1}^{r} \tau_{m-1} a_{n(j)}\right| < (2^{m-1} - 1)\varepsilon \qquad \text{for } 1 \leq r \leq n.$$

Since $\left||s| - \sum_{j=1}^{n} \sigma_m a_{n(j)}\right| < \varepsilon$ and $|a_{n(j)}| < \varepsilon$ for each j, there evidently is an r such that $\left|\lambda|s| - \sum_{j=1}^{r} \sigma_m a_{n(j)}\right| < \varepsilon$. For this r, then,

$$\left|\lambda s - \sum_{j=1}^{r} a_{n(j)}\right| < 2^{m-1}\varepsilon.$$

It follows that $\lambda s \in T_A$. Lemma 4 is proved.

LEMMA 5. *Let* A *and* B *be series in Euclidean spaces. Suppose that the following condition holds:*

(I) *For every* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* N *such that if* X *is a finite set of integers greater than* N, *and if* $\left|\sum_{n \in X} a_n\right| < \delta$, *then* $\left|\sum_{n \in X} b_n\right| < \varepsilon$.
Then for every $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $|\alpha_p - \alpha_q| < \delta$, *then* $|\beta_p - \beta_q| \leq \varepsilon$.

Remarks. Condition (I) implies the condition that A determines B. In fact, we shall see later that the two conditions are equivalent.

If A and B converge to zero, then by Lemma 4, $S_{A \times B}$ is a linear subspace. The conclusion of Lemma 5 implies that β_p is a continuous function of α_p. Since $S_{A \times B}$ is the graph of that function, it must be linear.

Proof of Lemma 5. Let $\varepsilon > 0$. Let δ and N be chosen corresponding to ε as in (I). Let p and q be arbitrary permutations such that A_p and A_q converge and $|\alpha_p - \alpha_q| < \delta$. We shall prove the lemma by showing that $|\beta_p - \beta_q| \leq \varepsilon$.

Choose $\eta > 0$ sufficiently small so that $|\alpha_p - \alpha_q| + 2\eta < \delta$. Let K be sufficiently large so that

$$\left|\alpha_p - \sum_{j=1}^{k} a_{p(j)}\right| < \eta \quad \text{and} \quad \left|\beta_p - \sum_{j=1}^{k} b_{p(j)}\right| < \eta \qquad \text{for } k \geq K,$$

and so that the set $Y = \{p(j): 1 \leq j \leq K\}$ contains all the integers less than or equal to N. Let L be sufficiently large so that

$$\left|\alpha_q - \sum_{j=1}^{r} a_{q(j)}\right| < \eta \quad \text{and} \quad \left|\beta_q - \sum_{j=1}^{r} b_{q(j)}\right| < \eta \qquad \text{for } r \geq L,$$

and so that the set $Z = \{q(j): 1 \leq j \leq L\}$ contains Y. Let

$$X = \{n: n \in Z \text{ and } n \notin Y\}.$$

Then $n \in X \Rightarrow n > N$, and

$$\left|\sum_{n \in X} a_n\right| < |\alpha_p - \alpha_q| + 2\eta < \delta,$$

and therefore $\left|\sum_{n\in X} b_n\right| < \varepsilon$. Since $|\beta_p - \beta_q| < \left|\sum_{n\in X} b_n\right| + 2\eta$, it follows that $|\beta_p - \beta_q| < \varepsilon + 2\eta$. Since η may be arbitrarily small, we may conclude that $|\beta_p - \beta_q| \leq \varepsilon$. Lemma 5 is proved.

The next lemma allows us to show that if A does not determine B, then $S_{A\times B} = S_A \times S_B$.

LEMMA 6. *Let* A *and* B *be conditionally convergent series in* R^m *and* R, *respectively. Then* (NI) $\Rightarrow$ (II) $\Rightarrow$ (III), *where the numerals denote the conditions stated below. Note that* (NI) *is the negation of* (I).

(NI) *There exists* $\eta > 0$ *such that for every* $\delta > 0$ *and every integer* $N > 0$, *there is a finite set* X *of integers greater than* N *such that* $\left|\sum_{n\in X} a_n\right| < \delta$ *and* $\left|\sum_{n\in X} b_n\right| > \eta$.

(II) *There exists* $\eta > 0$ *such that for every* $\delta > 0$, *every integer* $N > 0$, *and for* $u = +1$ *or* -1, *there is a finite set* X *of integers greater than* N *such that* $\left|\sum_{n\in X} a_n\right| < \delta$ *and* $u\sum_{n\in X} b_n > \eta$.

(III) *If* $\delta > 0$, $\varepsilon > 0$, $t \neq 0$, *and* $N > 0$, *then there is a finite set* Y *of integers greater than* N *such that* $\left|\sum_{n\in Y} a_n\right| < \delta$ *and* $\left|\sum_{n\in Y} b_n - t\right| < \varepsilon$.

Proof that (NI) $\Rightarrow$ (II). Let η be as in (NI). Let δ, N, and u be given as in the hypothesis of (II). We may suppose that N is sufficiently large so that if $N < m < M$, then $\left|\sum_{n=m}^{M} a_n\right| < \delta/2$ and $\left|\sum_{n=m}^{M} b_n\right| < \eta$. By applying (NI) twice, with appropriate choices of the parameters, we may find disjoint finite sets X_1 and X_2 of integers greater than N such that for i = 1 and 2, $\left|\sum_{n\in X_i} a_n\right| < \delta/4$ and $\left|\sum_{n\in X_i} b_n\right| > \eta$. If the two sums $\sum_{n\in X_i} b_n$ have opposite signs, the conclusion of (II) is satisfied by one of the sets X_i. Otherwise, let m be the minimum of the integers in $X = X_1 \cup X_2$, and let M be the maximum. Let

$$Y = \{n: m \leq n \leq M \text{ and } n \notin X\}.$$

Then $\left|\sum_{n\in X} a_n\right| < \delta/2$ and $\left|\sum_{n\in Y} a_n\right| < \delta$. Since $\left|\sum_{n\in X\cup Y} b_n\right| < \eta$ and $\left|\sum_{n\in X} b_n\right| > 2\eta$, we know that $\left|\sum_{n\in Y} b_n\right| > \eta$ and that $\sum_{n\in Y} b_n$ has the opposite sign from $\sum_{n\in X} b_n$, so that Y satisfies the conclusion of (II) if X does not.

Proof that (II) $\Rightarrow$ (III). Let δ, ε, t, and N be given. Let $\delta' < \delta/(2^m - 1)$. We may suppose that N is sufficiently large so that $|a_n| < \delta'$ and $|b_n| < \varepsilon$ whenever $n > N$. By repeated applications of (II), we may with appropriate choices of the parameters obtain a finite set X of integers greater than N such that $\left|\sum_{n\in X} a_n\right| < \delta'$ and $\sum_{n\in X} b_n > t$ (if t is positive) or $\sum_{n\in X} b_n < t$ (if t is negative). By Lemma 1, there exists an enumeration $X = \{n(j)\}_{j=1}^{J}$ such that $\left|\sum_{j=1}^{k} a_{n(j)}\right| < \delta$ for $1 \leq k \leq J$. Let k be the smallest integer such that $\left|\sum_{j=1}^{k} b_{n(j)}\right| > t$. Then $\sum_{j=1}^{k} b_{n(j)}$ evidently differs from t by no more than ε, and $\left|\sum_{j=1}^{k} a_{n(j)}\right| < \delta$. Let $Y = \{n(j): 1 \leq j \leq k\}$, and (III) is proved.

The proof of Lemma 6 is complete.

LEMMA 7. *Let* A *and* B *be conditionally convergent series in* R^m *and* R, *respectively, such that* A *does not determine* B. *Then* $S_{A\times B} = S_A \times S_B$.

Proof. Let $\alpha \in S_A$, $\beta \in S_B$. It suffices to prove that $\gamma \in T_C$, where $\gamma = (\alpha, \beta)$ and $C = A \times B$. Let $\varepsilon > 0$ and N be arbitrary. Since A does not determine B, (NI) holds, and hence (II) and (III) hold. Since $\alpha \in T_A$, there is a finite set Z of integers greater than N such that $\left|\alpha - \sum_{n\in Z} a_n\right| < \varepsilon/3$. Applying (III) with $t = \beta - \sum_{n\in Z} a_n$, one obtains a finite set Y of integers greater than N such that $Y \cap Z = \emptyset$, $\left|\sum_{n\in Y} a_n\right| < \varepsilon/3$, and $\left|\sum_{n\in Y} b_n - \beta + \sum_{n\in Z} b_n\right| < \varepsilon/3$. Then $X = Y \cup Z$ contains only integers greater than N, and $\left|\gamma - \sum_{n\in X} c_n\right| < \varepsilon$. Lemma 7 is proved.

LEMMA 8. *Let* A *and* B *be conditionally convergent series in* R^m *and* R, *respectively. Then* A *determines* B *if and only if* (I) *holds.*

Proof. The "if" part is clear. It remains to show that if (NI) holds, then there is a permutation p such that A_p converges but B_p does not. Let $s \in T_A$. Let q be a permutation defined as in the proof of Lemma 3, except that in Step 3 of that procedure, the choice of $m(k + 1)$ is further restricted so that $\sum_{n=1}^{m(k+1)} b_n$ is close to $(-1)^k$ (this is possible, in view of (III)). Then define p as in the proof of Lemma 2, and A_p will converge, whereas the partial sums $\sum_{j=i}^{n(i)} b_{p(j)}$ will oscillate. Lemma 8 is proved.

LEMMA 9. *Let* A *and* B *be conditionally convergent series in* R^m *and* R, *respectively, each with zero sum. If* A *determines* B, *then there is a surjective linear map* $L: R^m \to R$ *such that* $S_{A\times B} = \{(x, L(x)): x \in S_A\}$.

Proof. Let r be the integer between 1 and m such that $\tau_r A$ determines B but $\tau_{r-1} A$ does not. By Lemmas 8 and 5, there is a linear map $L: \tau_r R^m \to R$ such that

$$S_{\tau_r A\times B} = \{(x, L(x)): x \in S_{\tau_r A}\}.$$

In other words

$$S_{A\times B} = \{(x, L \circ \tau_r(x)): x \in S_A\}.$$

All that needs to be proved is that L is surjective, that is, $\sigma_{m+1} S_{A\times B} = R$. We may suppose that $r = m$, so that $L \circ \tau_r = L$. Let

$$C = (\tau_{m-1} A) \times B = (\tau_{m-1} + \sigma_{m+1})(A \times B).$$

Since $\tau_{m-1} A$ does not determine B, we see that $\dim S_C = 1 + \dim S_{\tau_{m-1} A}$ and $\sigma_{m+1} S_C = R$. Now $\tau_{m-1} A$ does not determine $\sigma_m A$, because if it did, then it would also determine B. Therefore

$$\dim S_{\tau_{m-1} A} = (\dim S_A) - 1 = (\dim S_{A\times B}) - 1.$$

Therefore $\dim S_C = \dim S_{A\times B}$, and hence C determines $\sigma_m A$. Hence there is a linear map $M: (\tau_{m-1} + \sigma_{m+1}) R^{m+1} \to \sigma_m R^{m+1}$ such that

$$S_{A\times B} = \{(u, v, w) \in (\tau_{m-1} R^{m+1}) \times (\sigma_m R^{m+1}) \times (\sigma_{m+1} R^{m+1}):$$
$$(u, w) \in S_C \text{ and } v = M(u, w)\}.$$

Since $\sigma_{m+1} S_C = R$, evidently $\sigma_{m+1} S_{A\times B} = R$. Lemma 9 is proved.

Proof of the Proposition. If A determines B, let L be the linear map given by Lemma 9. Let D denote the series $\sum_{k=1}^{\infty} (b_k - L(a_k))$. Since the series A determines B, it also determines D. In fact, for every p such that A_p converges, D_p converges to $\beta_p - L(\alpha_p)$, which always equals zero. Therefore $S_{A\times D} = S_A \times \{0\}$. If the convergence of D were conditional, then by Lemma 9, $\sigma_{m+1} S_{A\times D}$ would be R and not $\{0\}$. Therefore D converges absolutely.

If A does not determine B, then by Lemma 7, $S_{A\times B} = S_A \times S_B$. Therefore, if $\alpha_p \in S_A$ and $\beta_r \in S_B$, we know that there is a permutation q such that $\alpha_q = \alpha_p$ and $\beta_q = \beta_r$. We shall show that for every $\varepsilon > 0$, if we take k_0 sufficiently large, then we can modify the definition of q in a finite number of places so that $q(k) = p(k)$ for $k \leq k_0$ and

(1) $$\left|\sum_{k=k_0}^{r} a_{q(k)}\right| < \varepsilon \qquad \text{for all } r \geq k_0.$$

Then of course, A_q and B_q will still converge to α_p and β_r, respectively.

Given $\varepsilon > 0$, let $\varepsilon' = \varepsilon/(2^m - 1)$. Let k_0 be sufficiently large so that

(2) $$|a_{p(r)}| < \varepsilon'/2 \qquad \text{for all } r \geq k_0,$$

(3) $$\left|\sum_{k=k_0}^{\infty} a_{p(k)}\right| < \varepsilon'/4 .$$

Modify the definition of q(k) for a finite number of values of k, so that $q(k) = p(k)$ for $k \leq k_0$. For a sufficiently large $k_1 > k_0$,

(4) $$\left|\sum_{k=k_1}^{r} a_{q(k)}\right| < \varepsilon'/4 \qquad \text{for all } r \geq k_1.$$

By (2), $|a_{q(r)}| < \varepsilon'/2$ for all $r \geq k_0$; by (3) and (4), $\left|\sum_{k=k_0}^{k_1-1} a_{q(k)}\right| < \varepsilon'/2$. Therefore, by Lemma 1, q(k) may be redefined for $k_0 \leq k < k_1$ so that

(5) $$\left|\sum_{k=k_0}^{r} a_{q(k)}\right| < \varepsilon/2 \qquad \text{for } k_0 \leq r < k_1 .$$

Now (1) follows from (4) and (5). The proposition is proved.

3. NULL SEQUENCES IN R^∞

Here again, R^∞ denotes the countably infinite product of lines, with the product topology.

THEOREM 2. *Let* $\{a_k\}_{k=1}^\infty$ *be a sequence in* R^∞ *such that* $\lim_{k\to\infty} a_k = 0$. *Then there exists a sequence* $\{\varepsilon_k\}_{k=1}^\infty$, *with each* ε_k *equal to* $+1$ *or* -1, *such that* $\sum_{k=1}^\infty \varepsilon_k a_k$ *converges.*

The finite-dimensional version of this problem is taken care of by the following lemma, which is a simple and special case of results that appear in [3].

It will be convenient to use the ℓ^∞-norm in R^m. For $x = (x_1, \cdots, x_m) \in R^m$, $|x|$ will mean $\max\{|x_j|: 1 \leq j \leq m\}$.

LEMMA 10. *For every positive integer* m, *there is a constant* C_m *such that if* $\{a_k\}_{k=1}^\infty$ *is a sequence in* R^m *and* $|a_k| \leq r$ *for all* k, *then there exists a sequence* $\{\eta_k\}$, *with range* $\{-1, +1\}$, *such that* $\left|\sum_{k=1}^n \eta_k a_k\right| \leq C_m r$ *for all* n.

Proof of Theorem 2. The desired sequence $\{\varepsilon_k\}$ may be obtained by an inductive procedure. At the jth step, ε_k will be defined for $k(j) \leq k < k(j+1)$, where $\{k(j)\}_{j=0}^\infty$ is defined as follows. Let $k(0) = 1$. When $k(j-1)$ has been chosen, choose $k(j)$ to be an integer greater than $k(j-1)$ such that the quantity $r_j = \sup\{|\tau_j(a_k)|: k \geq k(j)\}$ is less than $2^{-j} C_j^{-1}$.

Let $\varepsilon_k = +1$ (say) for $k < k(1)$, and proceed to Step 1.

Step j (*for* j = 1, 2, $\cdots$). By Lemma 10, there is a sequence $\{\eta_{jk}\}_{k=k(j)}^\infty$ with range $\{-1, +1\}$, such that

$$\left|\sum_{k=k(j)}^{n} \eta_{jk}\, \tau_j(a_k)\right| \leq C_j r_j = 2^{-j} \qquad \text{for every } n > k(j).$$

Let $\varepsilon_k = \eta_{jk}$ for $k(j) \leq k < k(j+1)$.

The procedure is completely described. For each $j \geq 1$,

$$\left|\sum_{k=k(j)}^{n} \varepsilon_k\, \tau_j(a_k)\right| \leq 2^{-j+1} \qquad \text{for every } n > k(j).$$

Therefore $\sum \varepsilon_k a_k$ converges in R^∞. The theorem is proved.

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