THERE EXIST NONREFLEXIVE INFLATIONS

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1. INTRODUCTION

Let H be a complex Hilbert space, and let B(H) be the algebra of bounded linear operators on H. If U is a subalgebra of B(H), then Lat U represents the set of closed subspaces of H invariant under every member of U. If F is any set of closed subspaces of H, then Alg F is the algebra of bounded linear operators that leave invariant every member of F.

It is obvious that if U is a weakly closed subalgebra of B(H), and if it contains the identity operator, then $U \subseteq \text{Alg Lat } U$.

Following P. R. Halmos, we say U is *reflexive* if U = Alg Lat U. Sufficient conditions for an algebra of operators to be reflexive were given in [9], [4], [6], [1], and other papers. Most results are obtained by means of techniques developed by W. B. Arveson [2] and D. E. Sarason [9]. It should be pointed out that the problem of classifying all reflexive algebras includes various generalizations of the invariant-subspace problem (see [7], for example).

An algebra of operators $\mathcal{S}$ on H is an *n-inflation* if there exist a Hilbert space K, a subalgebra $U \subset B(K)$, and an integer n $(1 \leq n < \infty)$ such that

$$H = \sum_{i=1}^{n} \oplus K \quad \text{and} \quad \mathcal{S} = U^{(n)} = \left\{ \sum_{i=1}^{n} \oplus Ai \text{ with } Ai = A \in U \right\}.$$

In [8], P. Rosenthal raised the question whether every 2-inflation is reflexive. In this paper, we show that there exist 2-inflations on an infinite-dimensional Hilbert space that are not reflexive. For algebras generated by more than one operator, the answer is still unknown even in the finite-dimensional case.

I would like to thank my teacher Professor Peter Rosenthal for many valuable discussions with respect to the results of this paper. The techniques used in the proof of Theorem 1 were discovered by him and H. Radjavi [7].

2. PRELIMINARIES

By an *operator algebra* we shall mean a weakly closed subalgebra of B(H) that contains the identity operator.

Let U be an operator algebra on H. If U is reflexive, then so is $U^{(2)}$. For suppose C is an operator on $H^{(2)}$ such that $\text{Lat } U^{(2)} \subset \text{Lat } C$. Since $H \oplus \{0\}$, $\{0\} \oplus H$, and $\{\langle x, x\rangle : x \in H\}$ are all in Lat $U^{(2)}$, it follows that $C = B \oplus B$ for some B on H. Therefore $\text{Lat } U^{(2)} \subset \text{Lat } B^{(2)}$ implies $\text{Lat } U \subset \text{Lat } B$, and, since U is reflexive, $B \in U$.

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However, in general it is quite possible for $U^{(2)}$ to be reflexive while U is not. Suppose that T is a unicellular operator on a finite-dimensional Hilbert space, and let U be the algebra generated by T and I. Then $U^{(2)}$ is reflexive (this follows from a result of L. Brickman and P. Fillmore [3]), while it is easy to see that U is not reflexive.

3. THE MAIN RESULT

Arveson [2] has shown that there exist operator algebras on an infinite-dimensional Hilbert space that contain a maximal abelian self-adjoint algebra and are not reflexive. The existence of nonreflexive 2-inflations will follow from this and the following result.

THEOREM. *Let* U *be an operator algebra that contains a maximal abelian self-adjoint algebra. Then* U *is reflexive if and only if* $U^{(2)}$ *is reflexive.*

It has been shown that reflexivity of U implies the reflexivity of $U^{(2)}$. To prove the opposite assertion, a series of lemmas is necessary. For the proofs of Lemmas 3 and 4, we refer the reader to [7].

LEMMA 1. *Let* U *be an operator algebra and* P *a projection in* U. *If* $M \in \text{Lat } U$, *then* $PM \in \text{Lat } PUP$.

Proof. Suppose $M \in \text{Lat } U$. Since $P \in U$, we can assert that $M \in \text{Lat } PUP$. Also, since P is a projection, M reduces P and PM is closed. If $x \in PM$, then $APx = y \in M$ and $PAPx = Py \in PM$ for all $A \in U$.

LEMMA 2. *Suppose* $\text{Lat } U \subset \text{Lat } B$. *If* P *and* Q *are projections, then* $\text{Lat } PUQ \subset \text{Lat } PBQ$.

Proof. Let $M \in \text{Lat } PUQ$, and suppose $x \in M$ and $y \in M^{\perp}$ are chosen arbitrarily. We show that $(PBQx, y) = 0$.

Now $(PAQx, y) = 0$ for all $A \in U$. Thus $(AQx, Py) = 0$, and $\{AQx: A \in U\}$ is orthogonal to Py. Let N be the closure of $\{AQx: A \in U\}$. Then $N \in \text{Lat } U \subset \text{Lat } B$. Since $I \in U$, we see that $Qx \in N$. Therefore $BQx \in N$ and $(BQx, Py) = (PBQx, y) = 0$.

LEMMA 3. *Let* T *be a linear transformation (not necessarily bounded) that commutes with a maximal abelian self-adjoint algebra* R, *and let* $\mathcal{S}$ *be a strong basic neighborhood of the identity. Then there exists* $P \in \mathcal{S} \cap R$ *such that* $PTP \in R$.

LEMMA 4. *Let* M *be a subspace of* H, *and let* U *be an operator algebra. If in every strong neighborhood* $\mathcal{S}$ *of the identity there exists a projection* P *commuting with the projection onto* M *such that* $PM \in \text{Lat } PUP$, *then* $M \in \text{Lat } U$.

LEMMA 5. *Let* T *be a normal operator. Suppose* U *is an operator algebra and* B *is an operator such that* $\text{Lat } U \subset \text{Lat } B$. *If* $AT = TA$ *for all* $A \in U$, *then* $BT = TB$.

Proof. Since $AT = TA$ for all $A \in U$, each $A \in U$ commutes with every spectral projection of T. Since $\text{Lat } U \subset \text{Lat } B$ implies that B commutes with every spectral projection of T, we see that $BT = TB$.

Proof of the theorem. Suppose $U^{(2)}$ is reflexive. To show that U is reflexive, it is enough to show that if B is an operator such that $\text{Lat } U \subset \text{Lat } B$, then $\text{Lat } U^{(2)} \subset \text{Lat } B^{(2)}$.

Suppose $M \in \text{Lat } U^{(2)}$. We consider two cases.

Case 1. $M \cap (\{0\} \oplus H) = \{0\} \oplus \{0\}$; that is, $\langle 0, y \rangle \in M$ implies $y = 0$. Because M is a linear subspace, the second coordinate of a vector in M is linearly determined by the first coordinate. Thus, there exists a linear transformation T (possibly unbounded) such that

$$M = \{\langle x, Tx \rangle : x \in D\},$$

where D is the domain of T.

$M \in \text{Lat } U^{(2)}$ implies $AD \subset D$ and $AT = TA$ for all $A \in U$. In particular, every member of R commutes with T. Thus it follows from Lemma 3 that for each basic strong neighborhood $\mathcal{S}$ of the identity there exists a projection $P \in \mathcal{S} \cap R$ such that PTP is normal.

The operator $P^{(2)}$ is in every basic strong neighborhood of $I^{(2)}$. For let

$$\mathcal{L} = \{C \in B(H^{(2)}) : \|Cy_i - y_i\| < \varepsilon \text{ for } i = 1, \cdots, m\},$$

where $y_i = \langle x_1^i, x_2^i \rangle$, and let

$$\mathcal{S} = \{D \in B(H) : \|Dx_j^i - x_j^i\| < \varepsilon/2 \text{ for } i = 1, \cdots, m \text{ and } j = 1, 2\}.$$

Then

$$\|P^{(2)} y_i - y_i\| = \|\langle Px_1^i - x_1^i, Px_2^i - x_2^i \rangle\|$$
$$\leq \|Px_1^i - x_1^i\| + \|Px_2^i - x_2^i\| < 2\varepsilon/2 = \varepsilon.$$

Thus $P^{(2)} \in \mathcal{L}$.

Now $P^{(2)}M \in \text{Lat } P^{(2)} U^{(2)} P^{(2)}$, and by Lemma 4, it suffices to show that $P^{(2)} M \in \text{Lat } P^{(2)} B^{(2)} P^{(2)}$.

Now $P^{(2)} M = \{\langle Px, PTPx \rangle : x \in PD\}$, where $PTP \in R$ and

$$P^{(2)} M \in \text{Lat } P^{(2)} U^{(2)} P^{(2)}$$

implies $(PAP)(PTP) = (PTP)(PAP)$ for all $A \in U$. Thus, by Lemma 4, $(PTP)(PBP) = (PBP)(PTP)$ and $P^{(2)}M \in \text{Lat } P^{(2)}B^{(2)}P^{(2)}$.

Case 2. The intersection of M with $\{0\} \oplus H$ does not consist of the zero vector alone. Let $N = \{\langle 0, y \rangle \in M\}$. Then $N \in \text{Lat } U^{(2)} \subset \text{Lat } B^{(2)}$. Let $M' = M \ominus N$. Then $M' \cap (\{0\} \oplus H) = \{0\} \oplus \{0\}$ and $M' = \{\langle x, Tx \rangle : x \in D\}$, where T is again a linear transformation (possibly unbounded). Since R is self-adjoint, R reduces N, and therefore $M' \in \text{Lat } R$. Thus T commutes with every member of R. By Lemma 3, we can find $P \in \mathcal{S} \cap R$ such that PTP is normal. Thus it again suffices to show that $P^{(2)} M \subseteq \text{Lat } P^{(2)} B^{(2)} P^{(2)}$.

Since $P^{(2)}$ leaves M' and N invariant, it follows that

$$P^{(2)} M = P^{(2)} M' \oplus P^{(2)} N = \{\langle Px, PTPx \rangle : x \in PD\} \oplus \{\langle 0, Py \rangle\}.$$

For the sake of convenience, we shall henceforth omit the letter P, with the understanding that M will really mean $P^{(2)} M$, and so forth. Then

$$M = \{\langle x, Tx\rangle : x \in E_2\} \oplus \{\langle 0, y\rangle : y \in E_1\},$$

where T is a normal operator and $E_1, E_2 \in \text{Lat } U$.

If $AT = TA$ for all $A \in U$, Lemma 5 implies $BT = TB$, and there is nothing more to prove. Therefore we assume that there exists some $A \in U$ such that $AT \neq TA$. Then, if $\langle x, Tx\rangle \in M$,

$$A^{(2)}\langle x, Tx\rangle = \langle Ax, TAx\rangle + \langle 0, (AT - TA)x\rangle,$$

where $\langle 0, (AT - TA)x\rangle \in N$.

Since $T \in R$, $(AT - TA) \in U$ and $(AT - TA)x \in E_2$. Therefore $(AT - TA)x \in E_1 \cap E_2$. Let $F = E_1 \cap E_2$. Since $F \in \text{Lat } U$ and $T \in R$, F reduces T. Observe that in $M' \oplus N$ we can write

$$M' = \{\langle x, Tx\rangle : x \in E_2 \ominus F\} \oplus \{\langle x, Tx\rangle : x \in F\},$$

$$N = \{\langle 0, y\rangle : y \in E_1 \ominus F\} \oplus \{\langle 0, y\rangle : y \in F\}.$$

An examination of the second direct summands of M' and N shows that $(Tx, y) = 0$ for all $x, y \in F$. Thus $T = 0$ on F, and

$$M = \{\langle x, Tx\rangle : x \in E_2 \ominus F\} \oplus \{\langle 0, y\rangle : y \in E_1 \ominus F\} \oplus F^{(2)}.$$

Let Q be the projection onto $F^{\perp}$. Since $F^{(2)} \in \text{Lat } B^{(2)}$, it is enough to show that $Q^{(2)}M \in \text{Lat } Q^{(2)}B^{(2)}Q^{(2)}$. Now

$$Q^{(2)}M = \{\langle x, Tx\rangle : x \in E_2 \ominus F\} \oplus \{\langle 0, y\rangle : y \in E_1 \ominus F\},$$

and it is clear that $Q^{(2)}M \in \text{Lat } Q^{(2)}U^{(2)}Q^{(2)}$. Since $(E_1 \ominus F) \cap (E_2 \ominus F) = 0$, it follows that $(QAQ)T = T(QAQ)$ for all $A \in U$. Thus, by Lemma 5, $(QBQ)T = T(QBQ)$, and the proof is complete.

COROLLARY. *There exists a* 2-*inflation that is not reflexive.*

4. REMARKS

1. Whether every singly generated 2-inflation is reflexive is not known. A positive answer would lead to a number of existence theorems for invariant subspaces. One example: If the algebra generated by an operator A and the identity is not maximal abelian, then A has an invariant subspace.

2. It is not known whether there exist 2-inflations on a finite-dimensional Hilbert space that are not reflexive.

3. A sufficient condition for a 2-inflation to be reflexive is given in [5].

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