FINITE GROUPS WITHOUT FIXED-POINT-FREE ACTIONS ON A DISK

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A group of homeomorphisms of a space X is said to be fixed-point-free (FPF) provided no point of X is stationary under the whole group action. A natural question is whether a finite group can have an FPF action on a disk. In [3], J. Greever showed that groups of order $p^n q$ or pqr cannot act FPF and simplicially on disks; in particular, this means that the smallest possible example must have order at least 36. Actually, by using the techniques developed in [3], we can easily rule out all groups of order $p^2 q^2$, as well as all groups of order $p^2 q^3$ with $p < q$. Also, with the exception of the alternating group A_5, no group of order $p^2 qr$ with $p < q < r$ or $q < r < p$ can act FPF and simplicially on a disk. Thus A_5 is the smallest possible FPF group of simplicial homeomorphisms that a disk admits; an example of such an action is constructed in [2].

To establish our results, we first observe that if G is a group of homeomorphisms of X and N is a normal subgroup of G, then the collection F_N of points stationary under N inherits in a natural way an action by G/N; the collection of points stationary under *this* action is precisely F_G. As in [3], this is especially useful in conjunction with the next three theorems. For proofs, see the cited sources.

THEOREM 1. *Suppose* f *is a periodic simplicial homeomorphism of* X, *of prime-power period, where* X *is a finite complex. Then* $L(f) = \chi(F_f)$. (See [1, p. 550]; L(f) stands for the Lefschetz number of f, and $\chi(K)$ stands for the Euler characteristic of the complex K.)

THEOREM 2. *Suppose* G *is a finite* p-*group of simplicial homeomorphisms of* X, *where* X *is a finite complex. Then* $\chi(X) \equiv \chi(F_G) \pmod{p}$. (See [3, p. 165].)

THEOREM 3. *Let* G *and* X *satisfy the conditions of Theorem* 2, *and suppose further that* X *is homologically trivial* modulo p. *Then* F_G, *which by Theorem* 2 *is not empty, is also homologically trivial* modulo p. (See [3, p. 167].)

As an application, let G be a group of order $p^2 q^2$ acting simplicially on a disk. It is known [4, p. 146] that G must have a normal Sylow subgroup N; without loss of generality, we can assume that N is a p-group. By Theorem 3, F_N is homologically trivial modulo p, hence $\chi(F_N) = 1$. Theorem 2, applied to the q-group G/N, then implies that $\chi(F_G) \equiv 1 \pmod{q}$, so that G cannot be FPF.

A similar argument disposes of groups of order $p^2 q^3$ with $p < q$, since such groups must also have normal Sylow subgroups [4, p. 147], and we obtain the following theorem.

THEOREM 4. *Suppose that* G *is an* FPF *group of simplicial homeomorphisms of a disk. Then* G *cannot have order* $p^2 q^2$ *or* $p^2 q^3$ *with* $p < q$. *In particular,* G *cannot have order* 36.

Received August 9, 1973.

Michigan Math. J. 20 (1973).

Next we state a theorem that will be useful in the remaining applications.

THEOREM 5. *Let* G *be a finite group of simplicial homeomorphisms of a disk. Suppose that* $G \rhd H \rhd K \rhd 1$, *where* G/H *is a* p*-group,* H/K *is a cyclic* q*-group, and* K *is an* r*-group. Then* F_G *is not empty.*

Proof. F_K is homologically trivial modulo r, hence F_K has trivial rational homology. Therefore, $\chi(F_K) = 1$. Theorem 1, applied to a generator of H/K, implies that $\chi(F_H)$ is also 1. We obtain the desired result by applying Theorem 2 to the group G/H.

Let G be a group of order p^2qr acting simplicially on a disk, with $p < q < r$ and $G \neq A_5$. The case $p \geq 3$ is easiest to dispose of, for in such groups there is always a normal subgroup of order qr. To see this, let P be a p-Sylow subgroup of G, with normalizer N, centralizer C, and automorphism group A; recall that the order of A is either $p \cdot (p - 1)$ or $p \cdot (p + 1) \cdot (p - 1)^2$. Since P is Abelian, P is a subgroup of C, and hence the order of N/C must divide qr; since N/C is isomorphic to a subgroup of A, this order must divide $(p + 1) \cdot (p - 1)^2$ as well. Now $p \geq 3$ implies that $N = C$, and a theorem of Burnside [4, p. 137] guarantees that P has a normal complement H. Thus we obtain a normal series $G \rhd H \rhd Z_r \rhd 1$, with factor groups P, Z_q, and Z_r, and Theorem 5 implies that F_G is nonempty.

If $p = 2$ and $q \geq 5$, the argument above is still valid, and therefore we need only examine the case $p = 2$, $q = 3$. By Sylow's theorem, a group of order 12r must have a normal subgroup Z_r, unless $r = 5$ or $r = 11$; suppose, therefore, that $r \neq 5, 11$. Then G can be expressed as a semidirect product of Z_r by a subgroup H of order 12. Unless $H = A_4$, we obtain a normal series $G \rhd (Z_r \times_\theta Z_3) \rhd Z_r \rhd 1$, and Theorem 5 implies that F_G is nonempty. If $H = A_4$, then in the semidirect product $Z_r \times_\theta A_4$, the normal subgroup $Z_2 \oplus Z_2$ of A_4 must act trivially on Z_r, because the latter has a cyclic automorphism group. Hence we get a normal series $G \rhd [Z_r \times (Z_2 \oplus Z_2)] \rhd [Z_2 \oplus Z_2] \rhd 1$, and Theorem 5 applies.

It remains to treat groups of order 60 and 132. Let G be a group of 60 simplicial homeomorphisms of a disk, with $G \neq A_5$. Since G is solvable, there are three cases to consider.

Case 1. G contains a normal subgroup N of order 30. There are four groups of order 30, and each can be expressed as a semidirect product of Z_{15} by Z_2. Thus we obtain in this case a normal series $G \rhd N \rhd Z_{15} \rhd Z_5 \rhd 1$, and we may apply an argument similar to the proof of Theorem 5.

Case 2. G contains a normal subgroup N of order 20. There are five groups of order 20, all of which can be expressed as semidirect products of Z_5 by groups of order four. Therefore, we always have the series $G \rhd N \rhd Z_5 \rhd 1$. If the group N/Z_5 is cyclic, then we may appeal to Theorem 5. The only troublesome cases, then, occur when N is a semidirect product $Z_5 \times_\theta (Z_2 \oplus Z_2)$. The trivial product presents no problem, since we can write $G \rhd N \rhd (Z_2 \oplus Z_2) \rhd 1$. The nontrivial semidirect product can be described by the relations

$$a^5 = b^2 = c^2 = 1, \quad ba = a^4 b, \quad cb = bc, \quad ca = ac.$$

It is then a straightforward matter to verify that this group has 40 automorphisms, hence any semidirect product of this group by Z_3 must actually be a *direct* product. Since G is such a product, we see that G must contain an element of order 30, bringing us back to Case 1.

Case 3. G contains a normal subgroup N of order 12. Since no group of order 12 possesses an automorphism of order 5, it follows that G is just $N \times Z_5$. Unless N is A_4, N has an element of order 6, which again brings us back to Case 1. If N is A_4, we can write $G \rhd A_4 \rhd (Z_2 \oplus Z_2) \rhd 1$ and conclude that F_G is nonempty.

This takes care of all groups of order 60. Since the argument for order 132 is practically the same, we omit it. We have proved the following theorem.

THEOREM 6. *Let* G *be a group of simplicial homeomorphisms of a disk, of order* p^2qr *with* $p < q < r$, *and with* $G \neq A_5$. *Then* F_G *is not empty.*

We now turn to the cases $q < p < r$ and $q < r < p$. In either situation, a theorem of P. Hall [4, p. 228] implies that a group G of order p^2qr contains a subgroup H of order p^2r. Moreover, H must be normal because the index of H is the smallest prime divisor of the order of G. Unless $r \equiv 1 \pmod p$, we can write $G \rhd H \rhd K \rhd 1$, where K has order p^2, and then appeal to Theorem 5. We may therefore confine our search for FPF groups to the case $q < p < r$, with $r \equiv 1 \pmod p$. Since $H \rhd Z_r$ in such groups, we need only consider those groups whose p-Sylow subgroups are $Z_p \oplus Z_p$. Then the only remaining possible examples of FPF groups are of the form $[Z_r \times_\theta (Z_p \oplus Z_p)] \times_\phi Z_q$, where θ is nontrivial and Z_q does not act trivially on $Z_p \oplus Z_p$. It is a routine exercise to verify that groups of this form exist if and only if $p \equiv 1 \pmod q$ and $r \equiv 1 \pmod p$. If $r \not\equiv 1 \pmod q$, there is essentially one such group, which may be described by the relations

$$a^r = b^p = c^p = d^q = 1, \quad ba = ab, \quad cb = bc, \quad ca = a^\lambda c, \quad da = ad, \quad db = bd, \quad dc = c^\mu d,$$

where $\lambda^p \equiv 1 \pmod r$ and $\mu^q \equiv 1 \pmod p$. If $r \equiv 1 \pmod q$, there are q nonisomorphic groups; each may be obtained from the presentation above by replacing the relation $da = ad$ by a relation $da = a^\tau d$, where τ satisfies the condition $\tau^q \equiv 1 \pmod r$. We omit the details.

These groups, together with A_5, are precisely the groups of order p^2qr to which Theorem 5 can not be applied.

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