THE VOLUME OF A SMALL GEODESIC BALL OF A RIEMANNIAN MANIFOLD

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1. INTRODUCTION

Let M be an analytic Riemannian manifold. For $m \in M$, let $V_m(r)$ denote the volume of a geodesic ball centered at m with radius r. Then $V_m(r)$ can be expanded in a power series in r. In this note we compute the first five terms in this expansion. Our computation shows, for example, that if the Ricci scalar curvature of M is positive, then for small r, $V_m(r)$ is less than the corresponding function for Euclidean space. More generally if M is a C^∞ Riemannian manifold, we can compute the Taylor expansion of $V_m(r)$, although it may not converge.

In order to compute the Taylor expansion of $V_m(r)$, it is necessary to discuss general power series expansions of tensor fields in normal coordinates. In Section 2, we present a method for computing such expansions in modern notation. The coefficients of the power series expansions are polynomials in the covariant derivatives of the tensor fields and the curvature tensor.

Normal coordinate power series expressed in terms of the curvature operator occur implicitly in the classical literature of differential geometry, for example in books and papers by E. Cartan [5], L. P. Eisenhart [7], T. Y. Thomas [15], O. Veblen [16], and Veblen and Thomas [17]. Also, explicit formulas are given by A. Z. Petrov [11].

I know of several uses for power series expansions in normal coordinates, and probably there are many more. For example, such expansions have been used in the theory of harmonic spaces (see [12], for example) and in determining the asymptotic expansion for $\sum e^{-\lambda_i t}$, where the λ_i are the eigenvalues of the Laplacian of a compact Riemannian manifold.

Recently, P. Gilkey [8] also used them to give an analytic proof of the index theorem.

In Section 3, we use the results of Section 2 to compute the power series expansion of $V_m(r)$, and we derive several consequences of this expansion. Also, using the method of [1], we compute $V_m(r)$ explicitly for symmetric spaces of rank 1.

The coefficient of r^{n+4} in the expansion of $V_m(r)$ is especially interesting. (Here $n = \dim M$.) It is a quadratic invariant of $O(n)$. In Section 4, we compare it with other quadratic invariants arising from geometrical considerations. Notable among these are the conformal and spectral quadratic invariants and the 4-dimensional Gauss-Bonnet integrand. We discuss the linear independence among these and the quadratic invariant derived from $V_m(r)$ described above.

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2. POWER SERIES EXPANSIONS IN NORMAL COORDINATES

We assume that M is a C^∞ Riemannian manifold of dimension n with metric tensor $\langle\ ,\ \rangle$. Denote by $\mathfrak{X}(M)$ the C^∞ vector fields, and let ∇ and R be the Riemannian connection and curvature operator of M. Here ∇ and R are given by the formulas

$$(1)\qquad \begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle, \end{aligned}$$

$$(2)\qquad R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for X, Y, Z $\in \mathfrak{X}(M)$. Note that R is a tensor field but ∇ is not. We write $\nabla^p_{X\cdots X} = \nabla_X \cdots \nabla_X$.

Let $m \in M$, and let $(x_1, \cdots, x_n)$ be a normal coordinate system defined in a neighborhood $\tilde{U}_m$ of m with $x_1(m) = \cdots = x_n(m) = 0$. Denote by M_m the tangent space to M at m. In terms of the exponential map $\exp_m: U_m \to \tilde{U}_m$, any normal coordinate system of the type above is given by the formulas

$$x_j\left(\exp_m\left(\sum_{j=1}^n t_j u_j\right)\right) = t_j,$$

where $\{u_1, \cdots, u_n\}$ is an orthonormal basis of M_m.

We shall say that $X \in \mathfrak{X}(M)$ is a *coordinate vector field at* m if there exist constants $a_1, \cdots, a_n$ such that in a neighborhood of M we have the representation

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}.$$

Coordinate vector fields will be denoted by X, Y, $\cdots$, and their corresponding integral curves by $\alpha, \beta, \cdots$. We normalize so that $\alpha(0) = \beta(0) = m$. Thus $\alpha, \beta, \cdots$ are geodesics starting at m, and $\alpha'(t) = X_{\alpha(t)}$ wherever $\alpha(t)$ is defined.

LEMMA 2.1. *Let* X *and* Y *be coordinate vector fields, and let* α *be an integral curve of* X. *Then*

$$(3)\qquad (\nabla^p_{X\cdots X} X)_{\alpha(t)} = 0 \quad \textit{for } p = 1, 2, \cdots;$$

$$(4)\qquad (\nabla_X Y)_m = 0.$$

Proof. Since α is a geodesic, $(\nabla_X X)_{\alpha(t)} = 0$. From this, (3) follows by induction, because $(\nabla^p_{X\cdots X} X)_{\alpha(t)}$ depends only on the values of $X_{\alpha(t)}$ and $(\nabla^{p-1}_{X\cdots X} X)_\alpha$.

We always have the relation $\nabla_U V - \nabla_V U = [U, V]$, for all $U, V \in \mathfrak{X}(M)$. Since X and Y are coordinate vector fields, $[X, Y] = 0$. Hence $\nabla_X Y = \nabla_Y X$. Moreover, from (3) we see that $(\nabla_X Y)_m + (\nabla_Y X)_m = 0$. Hence (4) follows.

LEMMA 2.2. *For all* p, *we have the relations*

$$(\nabla^p_{YX\cdots X} X)_m = \cdots = (\nabla^p_{X\cdots XYX} X)_m,$$

where X *and* Y *are coordinate vector fields.*

Proof. Since $[X, Y] = 0$, equation (2) reduces to the formula

$$R_{XY} = -\nabla_X \nabla_Y + \nabla_Y \nabla_X. \tag{5}$$

Let

$$A_k = (\nabla^p_{X\cdots Y\cdots X} X)_m,$$

where Y occurs in the kth place. From (5) it follows that

$$A_k - A_{k-1} = (\nabla^{k-2}_{X\cdots X} R_{XY} \nabla^{p-k}_{X\cdots X} X)_m. \tag{6}$$

The right-hand side of (6) can be expanded in terms of the covariant derivatives of R at m. However, if $2 \leq k < p$, each term contains a factor of the form $(\nabla^i_{X\cdots X} X)_m$. By (3), each of these factors is zero. Hence $A_k = A_{k-1}$ for $2 \leq k < p$, and the lemma is proved.

LEMMA 2.3. *If* X *and* Y *are coordinate vector fields, then*

$$2(\nabla^p_{X\cdots XY} X)_m + (p-1)(\nabla^p_{X\cdots XYX} X)_m = 0 \qquad (p = 1, 2, \cdots).$$

Proof. This follows from the linearization of (3), the fact that $\nabla_X Y = \nabla_Y X$, and Lemma 2.2.

LEMMA 2.4. *If* X *and* Y *are coordinate vector fields, then*

$$(\nabla^p_{X\cdots X} Y)_m = -\left(\frac{p-1}{p+1}\right)(\nabla^{p-1}_{X\cdots X} R_{XY} X)_m \qquad (p = 1, 2, \cdots).$$

Proof. By (5) and Lemma 2.3,

$$(\nabla^{p-2}_{X\cdots X} R_{XY} X) = -(\nabla^p_{X\cdots XY} X)_m + (\nabla^p_{X\cdots XYX} X)_m = -\left(\frac{p+1}{p-1}\right)(\nabla^p_{X\cdots X} Y)_m.$$

In order to simplify the proof of the next theorems we introduce some simplified notation. Fix a point $m \in M$ and a coordinate vector field X at m. Define D and Q by

$$DY = \nabla_X Y, \qquad QY = R_{XY} X,$$

where Y is a coordinate vector field at m. Then

$$D^p Y = \nabla^p_{X\cdots X} Y \qquad \text{and} \qquad D^p(Q)Y = D^p(R)_{XY} X.$$

Also, put $D^\circ Y = Y$ and $D^\circ(Q) = Q$. We shall assume that all of these vector fields are evaluated at m. In this notation, Lemma 2.4 can be restated as

$$D^p Y = -\left(\frac{p-1}{p+1}\right) D^{p-2}(QY).$$

LEMMA 2.5. $$D^pY = -\left(\frac{p-1}{p+1}\right)D^{p-2}(Q)Y + \left(\frac{p-1}{p+1}\right)\sum_{i=1}^{[p/2]-1}\ \sum_{\substack{0 \le k_1+\cdots+k_i \le p-2i-2\\ k_j \ge 0}} (-1)^{i+1} \cdot \left(\prod_{j=1}^{i} t_p(k_1,\cdots,k_j)D^{k_j}(Q)\right)D^{p-k_1-\cdots-k_i-2i-2}(Q)Y,$$

where $t_p(k_1,\cdots,k_j) = \left(\dfrac{p-k_1-\cdots-k_j-2j-1}{p-k_1-\cdots-k_j-2j+1}\right)\cdot\dbinom{p-k_0-\cdots-k_{j-1}-2j}{k_j}$ *and* $k_0 = 0$.

Proof. One alternately applies Lemma 2.4 and the Leibniz rule

$$D^k(QY) = \sum_{s=0}^{k}\binom{k}{s}D^s(Q)D^{k-2}Y.$$

We omit the details.

The values of D^pY for $1 \le p \le 8$ are the following:

$$D^1Y = 0,\quad D^2Y = -\tfrac{1}{3}QY,\quad D^3Y = -\tfrac{1}{2}D(Q)Y,\quad D^4Y = \left\{-\tfrac{3}{5}D^2(Q) + \tfrac{1}{5}Q^2\right\}Y,$$

$$D^5Y = \left\{-\tfrac{2}{3}D^3(Q) + \tfrac{2}{3}D(Q)Q + \tfrac{1}{3}QD(Q)\right\}Y,$$

$$D^6Y = \left\{-\tfrac{5}{7}D^4(Q) + \tfrac{10}{7}D^2(Q)Q + \tfrac{3}{7}QD^2(Q) + \tfrac{10}{7}D(Q)^2 - \tfrac{1}{7}Q^3\right\}Y,$$

$$D^7Y = \left\{-\tfrac{3}{4}D^5(Q) + \tfrac{5}{2}D^3(Q)Q + \tfrac{1}{2}QD^3(Q) + \tfrac{15}{4}D^2(Q)D(Q) + \tfrac{9}{4}D(Q)D^2(Q)\right.$$
$$\left. - \tfrac{3}{4}D(Q)Q^2 - \tfrac{1}{2}QD(Q)Q - \tfrac{1}{4}Q^2D(Q)\right\}Y,$$

$$D^8Y = \left\{-\tfrac{7}{9}D^6(Q) + \tfrac{35}{9}D^4(Q)Q + \tfrac{5}{9}QD^4(Q) + \tfrac{70}{9}D^3(Q)D(Q) + \tfrac{28}{9}D(Q)D^3(Q)\right.$$
$$+ 7D^2(Q)^2 - \tfrac{7}{3}D^2(Q)Q^2 - \tfrac{10}{9}QD^2(Q)Q - \tfrac{1}{3}Q^2D^2(Q)$$
$$\left. - \tfrac{28}{9}D(Q)^2Q - \tfrac{14}{9}D(Q)QD(Q) - \tfrac{10}{9}QD(Q)^2 + \tfrac{1}{9}Q^4\right\}Y.$$

Although these formulas are special cases of Lemma 2.5, it is easier to prove them by using Lemma 2.4 directly.

Note that $D^p(Q)$ and D^pY are polynomials of degree $p+2$ in X. By standard procedures, they can be linearized. For example, in the previous notation the formula for D^2Y becomes

$$(\nabla^2_{XX}Y)_m = -\frac{1}{3}(R_{XY}X)_m.$$

We linearize this formula and obtain the relation

$$(\nabla_{WX}Y)_m + (\nabla_{XW}Y)_m = -\frac{1}{3}(R_{WY}X)_m - \frac{1}{3}(R_{XY}W)_m.$$

COROLLARY 2.6. *Suppose* M *is a symmetric space. Then*

$$D^{2p+1}\,Y = 0 \quad \textit{for } p = 0, 1, 2, \cdots,$$

$$D^{2p}\,Y = \frac{(-1)^p}{2p+1}\,Q^p \quad \textit{for } p = 1, 2, \cdots.$$

Assume now that M is an analytic Riemannian manifold. Let W be an analytic covariant tensor field defined in a neighborhood of m. Assume that $X_1, \cdots, X_n$ are coordinate vector fields that are orthonormal at m. Let $(x_1, \cdots, x_n)$ denote the corresponding normal coordinate system. We write

$$W(X_{\alpha_1}, \cdots, X_{\alpha_r}) = W_{\alpha_1 \cdots \alpha_r}.$$

Then we have a power series expansion

$$W_{\alpha_1 \cdots \alpha_r} = \sum_{k=0}^{\infty} \sum_{i_1,\cdots,i_k=1}^{n} \frac{1}{k!}(X_{i_1} \cdots X_{i_k} W_{\alpha_1 \cdots \alpha_r})(m)\, x_{i_1} \cdots x_{i_k}.$$

Here

$$(X^p W_{\alpha_1 \cdots \alpha_r})_m = \sum_{\substack{\nu_1+\cdots+\nu_{r+1}=p \\ \nu_i \geq 0}} \frac{p!}{\nu_1! \cdots \nu_{r+1}!} \tag{8}$$

$$\cdot \nabla^{\nu_{r+1}}_{X\cdots X}(W)(\nabla^{\nu_1}_{X\cdots X}X_{\alpha_1}, \cdots, \nabla^{\nu_r}_{X\cdots X}X_{\alpha_r})(m),$$

where X is a coordinate vector field and $(x_1, \cdots, x_n)$ is a normal coordinate system. The coefficients in the power series are symmetric in $X_{i_1} \cdots X_{i_k}$; therefore, we can determine these coefficients by linearizing the left-hand side of (8). On the other hand, Lemma 5 can be used to compute the right-hand side of (8).

By this method, it is theoretically possible to write down a general expression for the coefficients of the power series (7) in terms of covariant derivatives of W at m and the curvature operator of M at m. However, such an expression would be too cumbersome to be very useful. Instead, we compute the coefficients in (7) up to the fourth order. It will be clear that our method can be used to compute any other coefficient.

To simplify our notation, we now write

$$\nabla^p_{X_i \cdots X_j} = \nabla^p_{i \cdots j}, \quad \langle R_{X_iX_j}X_k, X_\ell \rangle = R_{ijk\ell}, \quad R_{ij} = \sum_{k=1}^{n} R_{ijik}.$$

THEOREM 2.7. *We have the expansion*

$$W_{\alpha_1\cdots\alpha_r} = W_{\alpha_1\cdots\alpha_r}(m) + \sum_{i=1}^{n} (\nabla_i W_{\alpha_1\cdots\alpha_r})(m)x_i$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\left\{\nabla^2_{ij}W_{\alpha_1\cdots\alpha_r} - \frac{1}{3}\sum_{s=1}^{n}\sum_{a=1}^{r} R_{i\alpha_a js}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right\}(m)x_i x_j$$

$$+\frac{1}{6}\sum_{i,j,k=1}^{n}\left\{\nabla^3_{ijk}W_{\alpha_1\cdots\alpha_r} - \sum_{s=1}^{n}\sum_{a=1}^{r} R_{i\alpha_a js}\nabla_k W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right.$$

$$\left.-\frac{1}{2}\sum_{s=1}^{n}\sum_{a=1}^{r}\nabla_i R_{j\alpha_a ks}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right\}(m)x_i x_j x_k$$

$$+\frac{1}{24}\sum_{i,j,k,\ell=1}^{n}\left\{\nabla^4_{ijk\ell}W_{\alpha_1\cdots\alpha_r} - 2\sum_{s=1}^{n}\sum_{a=1}^{r} R_{i\alpha_a js}\nabla^2_{k\ell}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right.$$

$$-2\sum_{s=1}^{n}\sum_{a=1}^{r}\nabla_i R_{j\alpha_a ks}\nabla_\ell W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}$$

$$-\frac{3}{5}\sum_{s=1}^{n}\sum_{a=1}^{r}\nabla^2_{ij}R_{k\alpha_a\ell s}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}$$

$$+\frac{1}{5}\sum_{s,t=1}^{n}\sum_{a=1}^{r} R_{i\alpha_a js}R_{ks\ell t}W_{\alpha_1\cdots\alpha_{a-1}t\alpha_{a+1}\cdots\alpha_r}$$

$$\left.+\frac{2}{3}\sum_{s,t=1}^{n}\sum_{\substack{a,b=1\\a<b}}^{r} R_{i\alpha_a js}R_{k\alpha_b\ell t}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_{b-1}t\alpha_{b+1}\cdots\alpha_r}\right\}(m)x_i x_j x_k x_\ell$$

$$+\cdots .$$

Proof. The theorem follows from Lemma 2.5. For example, we compute the cubic term in the expansion. For this it suffices to compute $X_i^3 W_{\alpha_1\cdots\alpha_r}(m)$ and to linearize. We obtain the function

$$X_i^3 W_{\alpha_1\cdots\alpha_r}(m) = \nabla^3_{iii}W_{\alpha_1\cdots\alpha_r}(m)$$

$$+\sum_{s=1}^{n}\sum_{a=1}^{r}\langle\nabla^3_{iii}X_{\alpha_a},\ X_s\rangle(m)W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}(m)$$

$$+3\sum_{s=1}^{n}\sum_{a=1}^{r}\left\langle \nabla_{ii}^{2}X_{\alpha_a}, X_s\right\rangle(m)\nabla_i W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}(m)$$

$$=\left\{\nabla_{iii}^{3}W_{\alpha_1\cdots\alpha_r}-\sum_{s=1}^{n}\sum_{a=1}^{r}R_{i\alpha_a is}\nabla_i W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right.$$

$$\left.-\frac{1}{2}\sum_{s=1}^{n}\sum_{a=1}^{r}\nabla_i R_{i\alpha_a is}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right\}(m).$$

Next we consider special cases of Theorem 2.7. Frequently, W is parallel, and then the expansion of Theorem 2.7 becomes considerably simpler.

COROLLARY 2.8. *In Theorem* 2.7, *assume that* W *is parallel* (*in other words, that all its covariant derivatives vanish*). *Then*

$$W_{\alpha_1\cdots\alpha_r}=W_{\alpha_1\cdots\alpha_r}(m)-\frac{1}{6}\sum_{i,j,s=1}^{n}\sum_{a=1}^{r}(R_{i\alpha_a js}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r})(m)x_i x_j$$

$$-\frac{1}{12}\sum_{i,j,k,s=1}^{n}\sum_{a=1}^{r}(\nabla_i R_{j\alpha_a ks}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r})(m)x_i x_j x_k$$

$$+\frac{1}{24}\sum_{i,j,k,\ell=1}^{n}\left\{-\frac{3}{5}\sum_{s=1}^{n}\sum_{a=1}^{r}\nabla_{ij}^{2}R_{k\alpha_a\ell s}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right.$$

$$+\frac{1}{5}\sum_{s,t=1}^{n}\sum_{a=1}^{r}R_{i\alpha_a js}R_{ks\ell t}W_{\alpha_1\cdots\alpha_{a-1}t\alpha_{a+1}\cdots\alpha_r}$$

$$\left.+\frac{2}{3}\sum_{s,t=1}^{n}\sum_{\substack{a,b=1\\a<b}}^{r}R_{i\alpha_a js}R_{k\alpha_b\ell t}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_{b-1}t\alpha_{b+1}\cdots\alpha_r}\right\}(m)x_i x_j x_k x_\ell$$

$$+\frac{1}{120}\sum_{i,j,k,\ell,h=1}^{n}\left\{-\frac{2}{3}\sum_{s=1}^{n}\sum_{a=1}^{r}\nabla_{ijk}^{3}R_{\ell\alpha_a hs}W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_r}\right.$$

$$+\sum_{s,t=1}^{n}\sum_{a=1}^{r}\left(\frac{1}{3}\nabla_i R_{j\alpha_a ks}R_{\ell sht}+\frac{2}{3}R_{i\alpha_a js}\nabla_k R_{\ell sht}\right)W_{\alpha_1\cdots\alpha_{a-1}t\alpha_{a+1}\cdots\alpha_r}$$

$$+\frac{5}{3}\sum_{s,t=1}^{n}\sum_{\substack{a,b=1\\a<b}}^{r}(\nabla_i R_{j\alpha_a ks}R_{\ell\alpha_b ht}+R_{i\alpha_a js}\nabla_k R_{\ell\alpha_b ht})$$

$$\left.\cdot W_{\alpha_1\cdots\alpha_{a-1}s\alpha_{a+1}\cdots\alpha_{b-1}t\alpha_{b+1}\cdots\alpha_r}\right\}(m)x_i x_j x_k x_\ell x_h+\cdots.$$

We apply Corollary 1.8 to the metric tensor. We write $g_{pq} = \langle X_p, X_q \rangle$. Then $g_{pq}(m) = \delta_{pq}$.

COROLLARY 2.9. *We have the expansion*

$$\begin{aligned} g_{pq} = \delta_{pq} &- \frac{1}{3}\sum_{i,j=1}^{n} R_{ipjq}(m)x_i x_j - \frac{1}{6}\sum_{i,j,k=1}^{n} \nabla_i R_{jpkq}(m)x_i x_j x_k \\ &+ \frac{1}{120}\sum_{i,j,k,\ell=1}^{n}\left\{-6\nabla^2_{ij}R_{kp\ell q} + \frac{16}{3}\sum_{s=1}^{n} R_{ipjs}R_{kq\ell s}\right\}(m)x_i x_j x_k x_\ell \\ &+ \frac{1}{90}\sum_{i,j,k,\ell,h=1}^{n}\left\{-\nabla^3_{ijk}R_{\ell phq} + 2\sum_{s=1}^{n}(\nabla_i R_{jpks}R_{\ell qhs} + \nabla_i R_{jqks}R_{\ell phs})\right\} \\ &\cdot (m)x_i x_j x_k x_\ell x_h + \cdots . \end{aligned}$$

As a second application of Corollary 1.8, we derive the power series expansion of the volume element of M.

Assume that M is orientable. Then up to sign there is a unique n-form ω such that $\omega(E_1, \cdots, E_n) = \pm 1$ for every orthonormal n-frame $\{E_1, \cdots, E_n\}$. Let $X_1, \cdots, X_n$ be coordinate vector fields that are orthonormal at the point m. Note that $X_1, \cdots, X_n$ need not be orthonormal at nearby points. We write

$$\omega_{1\cdots n} = \omega(X_1, \cdots, X_n).$$

COROLLARY 2.10. *We have the expansion*

$$\begin{aligned} \omega_{1\cdots n} = 1 &- \frac{1}{6}\sum_{i,j=1}^{n} R_{ij}(m)x_i x_j - \frac{1}{12}\sum_{i,j,k=1}^{n}\nabla_i R_{jk}(m)x_i x_j x_k \\ &+ \frac{1}{24}\sum_{i,j,k,\ell=1}^{n}\left\{-\frac{3}{5}\nabla^2_{ij}R_{k\ell} + \frac{1}{3}R_{ij}R_{k\ell} - \frac{2}{15}\sum_{a,b=1}^{n}R_{iajb}R_{ka\ell b}\right\}(m)x_i x_j x_k x_\ell \\ &+ \frac{1}{120}\sum_{i,j,k,\ell,h=1}^{n}\left\{-\frac{2}{3}\nabla^3_{ijk}R_{\ell h} + \frac{5}{3}\nabla_i R_{jk}R_{\ell h}\right. \\ &\qquad \left. - \frac{2}{3}\sum_{a,b=1}^{n}\nabla_i R_{jakb}R_{\ell ahb}\right\}(m)x_i x_j x_k x_\ell x_h \\ &+ \frac{1}{720}\sum_{i,j,k,\ell,h,g=1}^{n}\left\{-\frac{5}{7}\nabla^4_{ijk\ell}R_{hg} + 3\nabla^2_{ij}R_{k\ell}R_{hg} + \frac{5}{2}\nabla_i R_{jk}\nabla_\ell R_{hg}\right. \\ &\qquad - \frac{8}{7}\sum_{a,b=1}^{n}\nabla^2_{ij}R_{ka\ell b}R_{hagb} - \frac{5}{9}R_{ij}R_{k\ell}R_{hg} - \frac{15}{14}\sum_{a,b=1}^{n}\nabla_i R_{jakb}\nabla_\ell R_{hagb} \end{aligned}$$

$$- \frac{16}{63} \sum_{a,b,c=1}^{n} R_{iajb} R_{kb\ell c} R_{hcga}$$

$$+ \frac{2}{3} R_{ij} \sum_{a,b=1}^{n} R_{ka\ell b} R_{hagb} \Bigg\} (m)\, x_i x_j x_k x_\ell x_h x_g + \cdots .$$

COROLLARY 2.11. *We have the formula*

$$\omega^2_{1\cdots n} = 1 - \frac{1}{3} \sum_{i,j=1}^{n} R_{ij}(m)\, x_i x_j - \frac{1}{6} \sum_{i,j,k=1}^{n} \nabla_i R_{jk}(m)\, x_i x_j x_k$$

$$+ \frac{1}{24} \sum_{i,j,k,\ell=1}^{n} \left\{ -\frac{6}{5} \nabla^2_{ij} R_{k\ell} + \frac{4}{3} R_{ij} R_{k\ell} - \frac{4}{15} \sum_{a,b=1}^{n} R_{iajb} R_{ka\ell b} \right\} (m)\, x_i x_j x_k x_\ell$$

$$+ \frac{1}{120} \sum_{i,j,k,\ell,h=1}^{n} \left\{ -\frac{4}{3} \nabla^3_{ijk} R_{\ell h} + \frac{20}{3} \nabla_i R_{jk} R_{\ell h} - \frac{4}{3} \sum_{a,b=1}^{n} \nabla_i R_{jakb} R_{\ell a h b} \right\}$$

$$\cdot (m)\, x_i x_j x_k x_\ell x_h + \cdots .$$

Remark. In [2] and [13], the power series expansion of Corollary 2.11 is derived by a different method. Note that $\omega^2_{1\cdots n} = \det(g_{ij})$. In [2] and [13], the authors first expand g_{ij} in normal coordinates, and then they compute $\det(g_{ij})$ by multiplying the various power series. The two methods yield the same results.

3. POWER SERIES EXPANSIONS FOR VOLUME FUNCTIONS

Let M be an analytic Riemannian manifold, and let $r > 0$ be small enough so that $\exp_m$ is defined on a ball of radius r in the tangent space M_m. We put

$$S_m(r) = \text{volume of } \{\exp_m(x) \mid \|x\| = r\},$$

$$V_m(r) = \text{volume of } \{\exp_m(x) \mid \|x\| \le r\}.$$

Here we mean the $(n - 1)$-dimensional volume for $S_m(r)$ and the n-dimensional volume for $V_m(r)$. Also, we write

$$\tau(R) = \sum_{i=1}^{n} R_{ii}, \qquad \|R\|^2 = \sum_{i,j,k,\ell=1}^{n} R^2_{ijk\ell},$$

$$\|\rho(R)\|^2 = \sum_{i,j=1}^{n} R^2_{ij}, \qquad \Delta R = \text{Laplacian of } R = \sum_{i=1}^{n} \nabla^2_{ii}\, \tau(R).$$

THEOREM 3.1. *Write* $\alpha_n = 2\,\Gamma\left(\frac{1}{2}\right)^n \Gamma\left(\frac{n}{2}\right)^{-1}$. *Then*

$$S_m(r) = \alpha_n r^{n-1} \left\{ 1 - \frac{\tau(R)}{6n} r^2 + \frac{1}{360\,n(n+2)} (-3\,\|R\|^2 + 8\,\|\rho(R)\|^2 \right.$$

$$\left. + 5\tau(R)^2 - 18\Delta R) r^4 + O(r^6) \right\}_m ,$$

$$V_m(r) = \frac{\alpha_n r^n}{n} \left\{ 1 - \frac{\tau(R)}{6(n+2)} r^2 + \frac{1}{360\,(n+2)(n+4)} (-3\,\|R\|^2 + 8\,\|\rho(R)\|^2 \right.$$

$$\left. + 5\tau(R)^2 - 18\Delta R) r^4 + O(r^6) \right\}_m .$$

Proof. That $V_m(r) = \int_0^r S_m(t)\,dt$ is clear. Thus it suffices to compute $S_m(r)$. First we note that

$$S_m(r) = \int_{\exp_m(S^{n-1}(r))} i^*(\omega), \tag{9}$$

where ω is the volume element of M, $S^{n-1}(r)$ is the sphere of radius r in M_m, and i: $\exp_m(S^{n-1}(r)) \to M$ is the inclusion. Now consider the mapping

$$j: S^{n-1}(1) \to \exp_m(S^{n-1}(r))$$

defined by $j(u) = \exp_m(ru)$. Using j to change variables in (9), we obtain the equation

$$S_m(r) = \int_{S^{n-1}(1)} r^{n-1}\,\omega_{1\cdots n}(\exp_m(ru))\,du. \tag{10}$$

Let $\{x_1, \cdots, x_n\}$ be a normal coordinate system at m, and write $x_i = a_i r$ for $i = 1, \cdots, n$. Using Corollary 2.10, we can expand $\omega_{1\ldots n}(ru)$ in a power series in r, where the coefficients are homogeneous polynomials in the a_i. Thus

$$\omega_{1\ldots n}(\exp_m(ru)) = \sum_{p=0}^{\infty} \frac{\gamma_p}{p!} r^p,$$

where

$$\gamma_0 = 1, \quad \gamma_1 = 0, \quad \gamma_2 = -\frac{1}{3}\sum_{i,j=1}^{n} R_{ij} a_i a_j, \quad \gamma_3 = -\frac{1}{2}\sum_{i,j,k=1}^{n} \nabla_i R_{jk} a_i a_j a_k,$$

$$\gamma_4 = \sum_{i,j,k,\ell=1}^{n} \left\{ -\frac{3}{5}\nabla^2_{ij} R_{k\ell} + \frac{1}{3} R_{ij} R_{k\ell} - \frac{2}{15}\sum_{a,b=1}^{n} R_{iajb} R_{ka\ell b} \right\} a_i a_j a_k a_\ell, \quad \cdots .$$

We have simplified the notation by assuming that all the coefficients are evaluated at m. From (10), we obtain the formula

(11) $$S_m(r) = r^{n-1} \sum_{p=0}^{\infty} \frac{r^p}{p!} \int_{S^{n-1}(1)} \gamma_p \, du .$$

The computation of the integrals of the γ_i is the well-known moment problem. First note that

$$\int_{S^{n-1}(1)} \gamma_p \, du = 0 \qquad \text{for odd indices } p .$$

Furthermore,

$$\int_{S^{n-1}(1)} \gamma_0 \, du = \text{volume}\,(S^{n-1}(1)) = 2\,\Gamma\left(\frac{1}{2}\right)^n \Gamma\left(\frac{n}{2}\right)^{-1} .$$

Next,

$$\int_{S^{n-1}(1)} \gamma_2 \, du = - \sum_{i,j=1}^{n} \int_{S^{n-1}(1)} R_{ij} a_i a_j \, du$$

$$= - \sum_{i=1}^{n} R_{ii} \int_{S^{n-1}(1)} a_i^2 \, du = - \frac{2}{n} \Gamma\left(\frac{1}{2}\right)^n \Gamma\left(\frac{n}{2}\right)^{-1} \tau(R),$$

because $a_1^2 + \cdots + a_n^2 = 1$. To compute γ_4, we note that

$$\int_{S^{n-1}(1)} a_i^4 \, du = \frac{6}{n(n+2)} \Gamma\left(\frac{1}{2}\right)^n \Gamma\left(\frac{n}{2}\right)^{-1}$$

and

$$\int_{S^{n-1}(1)} a_i^2 a_j^2 \, du = \frac{2}{n(n+2)} \Gamma\left(\frac{1}{2}\right)^n \Gamma\left(\frac{n}{2}\right)^{-1} \qquad \text{for } i \neq j .$$

Write $\lambda_{ijk\ell} = - \frac{3}{5} \nabla_{ij}^2 R_{k\ell} + \frac{1}{3} R_{ij} R_{k\ell} - \frac{2}{15} \sum_{a,b=1}^{n} R_{iajb} R_{ka\ell b}$. Then

$$\frac{1}{\alpha_n} \int_{S^{n-1}(1)} \gamma_4 \, dV = \frac{1}{\alpha_n} \sum_{i,j,k,\ell=1}^{n} \lambda_{ijk\ell} \int a_i a_j a_k a_\ell \, du$$

$$= \frac{3}{n(n+2)} \sum_{i=1}^{n} \lambda_{iiii} + \frac{1}{n(n+2)} \sum_{i \neq j} (\lambda_{iijj} + \lambda_{ijij} + \lambda_{ijji})$$

$$= \frac{1}{n(n+2)} \sum_{i,j=1}^{n} (\lambda_{iijj} + \lambda_{ijij} + \lambda_{ijji})$$

$$= \frac{1}{n(n+2)} \sum_{i,j=1}^{n} \left\{ -\frac{3}{5} \nabla^2_{ii} R_{jj} - \frac{6}{5} \nabla^2_{ij} R_{ij} + \frac{1}{3} R_{ii} R_{jj} + \frac{2}{3} R^2_{ij} \right.$$

$$\left. - \frac{2}{15} \sum_{a,b=1}^{n} (R_{iaib} R_{jajb} + R^2_{iajb} + R_{iajb} R_{jaib}) \right\}$$

$$= \frac{1}{15\, n(n+2)} \{ -3 \|R\|^2 + 8 \|\rho(R)\|^2 + 5\tau(R) - 18 \Delta R \} .$$

The formula for $S_m(r)$ now follows from (11).

Note that for Euclidean space $\mathbb{R}^n$,

$$S_m(r) = \alpha_n r^{n-1} \quad \text{and} \quad V_m(r) = \frac{\alpha_n r^n}{n} .$$

Therefore we have the following consequence of Theorem 3.1.

COROLLARY 3.2. *Suppose* M *is an analytic Riemannian manifold with positive Ricci scalar curvature* $\tau(R)$ *at* m. *Then, for sufficiently small* $r > 0$,

$$S_m(r) < \alpha_n r^{n-1} \quad \textit{and} \quad V_m(r) < \frac{\alpha_n r^n}{n} . \tag{12}$$

If $\tau(R)$ *is negative, then for sufficiently small* $r > 0$,

$$S_m(r) > \alpha_n r^{n-1} \quad \textit{and} \quad V_m(r) > \frac{\alpha_n r^n}{n} .$$

Remark. Bishop [3] has proved that if M has nonnegative Ricci curvature everywhere, then (12) holds for all r less than or equal to the distance between m and its cut locus. Both the hypotheses and conclusion of Bishop's theorem are stronger than those of Corollary 3.2. (See also [4, p. 256].)

COROLLARY 3.3. *Suppose the Ricci curvature* $\rho(R)$ *vanishes at* m. *Then*

$$S_m(r) = \alpha_n r^{n-1} \left\{ 1 - \frac{\|R\|^2 r^4}{120\, n(n+2)} + O(r^6) \right\}_m ,$$

$$V_m(r) = \frac{\alpha_n r^n}{n} \left\{ 1 - \frac{\|R\|^2 r^4}{120\,(n+2)(n+4)} + O(r^6) \right\}_m .$$

Thus for manifolds with zero Ricci curvature, $S_m(r) \leq \alpha_n r^{n-1}$; for small r, the number $\|R\|^2$ measures how much $S_m(r)$ is actually less than $\alpha_n r^{n-1}$.

Examples. If M is a symmetric space of rank 1, we may compute the functions $S_m(r)$ and $V_m(r)$ explicitly. We shall now write these down for the sphere, complex projective space, quaternionic projective space, and the Cayley plane. We can find the corresponding formulas for the noncompact duals of these spaces by substituting sinh for sin and cosh for cos.

1. *The sphere* $S^n(1/\sqrt{\lambda})$ (with constant curvature λ):

$$S_m(r) = 2\,\Gamma\left(\frac{1}{2}\right)^n \Gamma\left(\frac{n}{2}\right)^{-1} \lambda^{-\frac{1}{2}(n-1)} \sin^{n-1}(\sqrt{\lambda}r).$$

2. *Complex projective space* CP^n (with constant holomorphic sectional curvature 4λ):

$$S_m(r) = \frac{2\pi^n}{(n-1)!\,\lambda^{n-\frac{1}{2}}} \sin^{2n-1}(\sqrt{\lambda}r)\cos(\sqrt{\lambda}r),$$

$$V_m(r) = \frac{\pi^n}{n!\,\lambda^n} \sin^{2n}(\sqrt{\lambda}r).$$

3. *Quaternionic projective space* QP^n (with maximum sectional curvature 4λ):

$$S_m(r) = \frac{\pi^{2n}}{4(2n-1)!\,\lambda^{2n-\frac{1}{2}}} \sin^3(2\sqrt{\lambda}r)\sin^{4n-4}(\sqrt{\lambda}r),$$

$$V_m(r) = \frac{\pi^{2n}}{(2n+1)!\,\lambda^{2n}} \sin^{4n}(\sqrt{\lambda}\,r)\ \{2n\cos^2(\sqrt{\lambda}r) + 1\}.$$

4. *The Cayley plane* Cay P^2 (with maximum sectional curvature 4λ):

$$S_m(r) = \frac{\pi^8}{7!\,2^6\lambda^{15/2}} \sin^7(2\sqrt{\lambda}r)\sin^8(\sqrt{\lambda}r),$$

$$V_m(r) = \frac{6\pi^8}{11!\,\lambda^8} \sin^{16}(\sqrt{\lambda}r)\ \{120\cos^6(\sqrt{\lambda}r) + 36\cos^4(\sqrt{\lambda}r) + 8\cos^2(\sqrt{\lambda}r) + 1\}.$$

These formulas may be computed by the method of [1].

4. QUADRATIC INVARIANTS OF CURVATURE OPERATORS

If V is an n-dimensional vector space with metric $\langle\ ,\ \rangle$, let $\Lambda^2(V)$ denote the space of 2-vectors. Then $\Lambda^2(V)$ has an induced metric. We denote by $\mathscr{R}(V)$ the space of symmetric linear operators on $\Lambda^2(V)$ that satisfy the first Bianchi identity. Then $\dim \mathscr{R}(V) = \frac{1}{12} n^2(n^2-1)$.

In this section, we consider the space $\mathscr{P}_2(V)$ of quadratic polynomials on $\mathscr{R}(V)$ that are invariant under the action of O(n). In [1], it is shown that $\dim \mathscr{P}_2(V) = 3$. In fact, a basis of $\mathscr{P}_2(V)$ consists of the quadratic polynomials $\|R\|^2$, $\|\rho(R)\|^2$, and $\tau(R)^2$ described in [2].

Another basis that arises naturally in the case $\dim V = 4$ is described in [6] and [9]. Let $G_{2,2}$ denote the submanifold of the unit sphere in $\Lambda^2(V)$ consisting of decomposable vectors; then $G_{2,2}$ is the Grassmann manifold of 2-planes in 4-space. The sectional curvature of a curvature tensor $R \in \mathscr{R}(V)$ is the differentiable function $K: G_{2,2} \to \mathbb{R}$ defined by $K(\xi) = \langle K\xi, \xi\rangle$. We also define $K^\perp: G_{2,2} \to \mathbb{R}$ by $K^\perp(\xi) = \langle R*\xi, *\xi\rangle$, where $*: \Lambda^2(V) \to \Lambda^2(V)$ is the star operator. Then the polynomials

$$\int_{G_{2,2}} K^2\, dV, \quad \int_{G_{2,2}} KK^{\perp}\, dV, \quad \left\{\int_{G_{2,2}} K\, dV\right\}^2$$

also form a basis of $\mathscr{P}_2(V)$. In fact, we have the following proposition.

PROPOSITION 4.1. *Let* $\dim V = 4$, *and write* $L = \operatorname{vol}(G_{2,2})$. *Then*

$$\tau(R)^2 = \frac{144}{L^2}\left\{\int_{G_{2,2}} K\, dV\right\}^2,$$

$$\|\rho(R)\|^2 = \frac{18}{L}\int_{G_{2,2}} (K^2 - KK^{\perp})\, dV + \frac{36}{L^2}\left\{\int_{G_{2,2}} K\, dV\right\}^2,$$

$$\|R\|^2 = \frac{24}{L}\int_{G_{2,2}} (4K^2 + KK^{\perp})\, dV - \frac{96}{L^2}\left\{\int_{G_{2,2}} K\, dV\right\}^2.$$

Hence $\int_{G_{2,2}} K^2\, dV$, $\int_{G_{2,2}} KK^{\perp}\, dV$, *and* $\left\{\int_{G_{2,2}} K\, dV\right\}^2$ *form a basis of* $\mathscr{P}_2(V)$.

We now define four somewhat more complicated polynomials in $\mathscr{P}_2(V)$. Each of them arises from geometric structure on a manifold.

Definitions. 1. The *volumal* quadratic polynomial is

$$V_2(R) = -3\|R\|^2 + 8\|\rho(R)\|^2 + 5\tau(R)^2 + 18\Delta R.$$

2. The *conformal* quadratic polynomial is

$$C_2(R) = \|R\|^2 - \frac{4}{n-2}\|\rho(R)\|^2 + \frac{2}{(n-1)(n-2)}\tau(R)^2.$$

3. The *spectral* quadratic polynomial is

$$S_2(R) = \frac{1}{360}\{2\|R\|^2 - 2\|\rho(R)\|^2 + 5\tau(R)^2 + 12\Delta R\}.$$

4. The 4-*dimensional Gauss-Bonnet integrand* is

$$X(R) = \{\|R\|^2 - 4\|\rho(R)\|^2 + \tau(R)^2\}.$$

The volumal quadratic form arises in Theorem 3.1. We now explain the role of the other polynomials.

PROPOSITION 4.2. *Under a conformal change of metric* $\langle\ ,\ \rangle \to e^{2\sigma}\langle\ ,\ \rangle$ *on a Riemannian manifold* M, *the conformal polynomial* $C_2(R)$ *is transformed into* $e^{-4\sigma} C_2(R)$. *Hence* $\int_M C_2(R)\, dM$ *is a conformal invariant. Furthermore,* $C_2(R) = \operatorname{tr}(W^2)$, *where* W *is the Weyl conformal tensor. Finally,* $C_2(R)$ *is the only quadratic polynomial on the space of curvature operators with the property of being invariant under* O(n) *and being transformed by a scalar under a conformal change of metric on* M.

This may be proved by some lengthy calculations (see [14]).

PROPOSITION 4.3. *Let* M *and* M' *be two compact Riemannian manifolds whose Laplacians have the same eigenvalues. Denote by* R *and* R' *the curvature operators of* M. *Then*

$$\int_M S_2(R)\,dM = \int_{M'} S_2(R')\,dM'.$$

For this, see [1], [2], [10]. Note that $\int_M \Delta R\,dM = 0$.

PROPOSITION 4.4. *Let* M *be a* 4-*dimensional compact Riemannian manifold. Then*

$$\chi(M) = \int_M X(R)\,dM,$$

where $\chi(M)$ *denotes the Euler characteristic of* M.

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