CONTINUITY PROPERTIES OF THE VISIBILITY FUNCTION

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Intuitively, the visibility function for a set E in R^n measures the n-dimensional volume of the part of E visible from a variable point of E. The purpose of this paper is to establish some continuity properties of this function. The function is not as well-behaved as one might expect: we present an example of a compact plane set K with locally connected boundary of measure zero and with a discontinuity of the visibility function at an interior point. However, if K is a compact plane set with a locally connected boundary and simply connected components, the function is continuous on the interior of K. As a main result we characterize the compact sets in R^n on which the visibility function is continuous.

1. PRELIMINARIES

Definition. The *visibility function* assigns to each point x of a fixed measurable set E in a Euclidean space R^n the Lebesgue outer measure of the set

$$S(x) = \{y: rx + (1 - r)y \in E \text{ for every } r \text{ in } [0, 1]\}.$$

In [1], the reader will find a more general discussion of the visibility function and its use in describing the relative convexity of a set. In this article, we characterize the compact sets whose visibility functions are continuous, and we present sufficient conditions for the continuity of the function on compact sets and on bounded open sets in the plane.

We need three theorems established in [1].

THEOREM 1. *If* $E \subset R^n$ *is open, then the visibility function associated with* E *is lower-semicontinuous.*

THEOREM 2. *Let* $K \subset R^n$ *be compact. Then the visibility function associated with* K *is upper-semicontinuous.*

THEOREM 3. *Let* K *be a compact set in* R^n. *If* $x \in K$, *the set of all endpoints of maximal segments in* S(x) *with one endpoint at* x *forms a measurable set and has measure zero.*

We use essentially the same notation as in [1]. Ordinary Lebesgue measure in R^n is denoted by m. By $B_r(x)$ we denote the closed r-ball about a point x, and by conv E the convex hull of E. The line segment joining x to y is denoted by xy, and L(x, y) will symbolize the line determined by x and y. The symbols int E, cl E, bd E, and E^c denote as usual the interior, closure, boundary, and complement of E. Finally, the interior of a set E relative to the smallest flat containing E is denoted by intv E, and when a fixed set E is under discussion, we shall denote the visibility function for the set by the letter v.

The standard technique used to establish the lower-semicontinuity (and, hence, the continuity) of the function v at a point x of a compact set K is to show that

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whenever $\{x_n\}$ is a convergent sequence in K with limit x, then all of S(x) except a subset of measure zero is contained in $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)$. It then follows that $v(x) \leq \liminf_{k \to \infty} v(x_k)$. Similarly, in the case of an open set we show that essentially $S(x) \supset \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S(x_n)$, to obtain the upper-semicontinuity of the function at x.

2. COMPACT SETS

To establish our characterization theorem, we shall use the following theorem of Dini.

THEOREM. *Let* $\{f_n\}$ *be a sequence of upper-semicontinuous, nonnegative functions defined on a compact set* K *in* R^n. *Suppose for each* x *in* K, *the sequence* $\{f_n(x)\}$ *converges monotonically to zero. Then* $\{f_n\}$ *converges uniformly to the zero function on* K.

Definition. Let K be a compact set in R^n. The ε-*parallel* body of K, denoted by $B_\varepsilon(K)$, is the compact set $\bigcup_{x \in K} B_\varepsilon(x)$.

THEOREM 4. *Let* K *be a compact set in* R^n, *and for each* $m \in Z^+$, *let* v_m *denote the visibility function for the* $1/m$-*parallel body of* K. *Then the visibility function* v *of* K *is continuous if and only if the sequence* $\{v_m \mid K\}$ *converges uniformly to* v.

Proof. Suppose v is discontinuous at $p \in K$. Since v is upper-semicontinuous at p, there exist $\varepsilon > 0$ and points $p_k \in K$ such that $\|p_k - p\| < 1/k$ and $v(p_k) < v(p) - \varepsilon$ for $k = 1, 2, \cdots$. Clearly, if $k > m$, then S(p) is a subset of the points that p_k sees via $B_{1/m}(K)$, so that $v_m(p_k) > v(p_k) + \varepsilon$. Since m is arbitrary, $\{v_m \mid K\}$ does not converge to v uniformly.

Conversely, we show that $\{v_m \mid K - v\}$ satisfies the hypothesis of Dini's Theorem. Since $v_m \mid K$ is upper-semicontinuous and v is continuous, $v_m \mid K - v$ is upper-semicontinuous. We claim that $\{v_m \mid K - v\}$ converges monotonically to zero, which is to say that for each $p \in K$, S(p) is the set of points seen by p via $B_{1/m}(K)$ for all m. Clearly, if $x \in S(p)$, then x is seen by p via $B_{1/m}(K)$ for all m; however, if x is seen by p via $B_{1/m}(K)$ for all m, then each point of xp must be a limit point of K, and therefore must belong to K. Since K is compact, the result follows. ∎

It is easy to construct examples of compact sets on which the visibility function is discontinuous. For example, if E is the union of two externally tangent closed discs in the plane, then the visibility function for E is discontinuous at the point of tangency. As we shall see, discontinuities may occur in the interior of a set even when the boundary does not seem pathological. We first establish a sharp planar result.

THEOREM 5. *Let* K *be a compact set in* R^2 *whose components are simply connected and whose boundary is locally connected. Then* v *is continuous on* int K.

Proof. Let $x \in$ int K. To show that v is lower-semicontinuous at x, we need only show that $m(S(x) \setminus \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)) = 0$ whenever $\{x_n\} \to x$.

If $x \in$ int K, we denote by C_1 the family of maximal segments in S(x) that have one endpoint at x and contain only one point of bd K, and by C_2 the remaining maximal segments in $S(x)$ with one endpoint at x. If $y \in S(x)$ is invisible from infinitely many points x_n, then $xy \not\subset \operatorname{int} K$. Hence, if y is not an endpoint of a segment in C_1, then y must belong to a segment in C_2. Since

$$\{y\colon y \text{ is an endpoint of a segment in } C_1\}$$

is a set of measure zero, we need only show that the number of segments in C_2 is countable.

If not, then for some $\varepsilon' > 0$ there exist infinitely many segments $\{a_n b_n\}$ such that

$$a_n \in \operatorname{bd} K, \quad b_n \in \operatorname{bd} K, \quad a_n \in xb_n, \quad b_n \in S(x), \quad \|a_n - b_n\| > \varepsilon'.$$

Passing to a subsequence, we may assume that $\{a_n b_n\}$ converges to a segment ab, where $\|a - b\| \geq \varepsilon'$ and $\{a, b\} \subset S(x) \cap \operatorname{bd} K$. Let $\varepsilon = \min\{\|a - x\|, \varepsilon'\}$. Let $N_b \subset B_{\varepsilon/4}(b)$ be a connected neighborhood of b in $\operatorname{bd} K$, and let $N_a \subset B_{\varepsilon/4}(a)$ be a connected neighborhood of a in $\operatorname{bd} K$. If n is large, then $\{a_n, a_{n+1}\} \subset N_a$ and $\{b_n, b_{n+1}\} \subset N_b$. By the definition of N_b, all points of the set $xb_n \cup xb_{n+1} \cup N_b$ belong to the same component C of K, and since C^c is connected and open, it follows that

$$\operatorname{int}\operatorname{conv}(x \cup b_n \cup b_{n+1}) \cap B_{\varepsilon/4}(a) \subset \operatorname{int} K.$$

It is clear that a_n and a_{n+1} cannot be connected by boundary points in N_a, and this yields a contradiction to our assumption that the number of segments in the family C_2 is uncountable. Thus

$$m\left(S(x) \setminus \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)\right) = 0,$$

so that

$$m(S(x)) \leq m\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)\right);$$

this yields the lower-semicontinuity of v at x. ■

The following two counterexamples arise from Cantor sets of positive measure. We form such a set in $[0, 2\pi]$ in the usual way by throwing out a countable collection of open intervals. At the nth stage, we throw out the union O_n, of 2^{n-1} centrally situated open intervals, each of length $2\pi/2 \cdot 3^n$. Let K_n denote the closed set remaining after the nth stage.

It is now convenient to represent points in the plane by their polar coordinates. Let E_0 denote the set

$$\{(r, \theta)\colon r \leq 1\} \cup \left\{(r, \theta)\colon 1 < r \leq 2,\ \theta \in \bigcap_{n=1}^{\infty} K_n\right\}.$$

This set is compact and simply connected, but its boundary is not locally connected. Since $E_0 \cap \{(r, \theta)\colon 1 < r \leq 2\}$ is nowhere dense, we see that if $0 < r < 1$ and

$0 \leq \theta < 2\pi$, the set $S(r, \theta)$ consists of the closed unit disk together with at most two of the line segments of E_0 outside of the disk. Because E_0 is starshaped, all of it is visible from the origin, and therefore the visibility function is discontinuous at the origin.

Let ρ_n denote 3/2 times the length of one of the closed intervals forming the set K_n. Clearly, $\rho_n \to 0$. For each set O_n let

$$O_n' = \{(r, \theta): \theta \in O_n,\ 1 < r < 1 + \sqrt{\rho_n}\}.$$

Let $E = \{(r, \theta): r \leq 2\} \setminus \bigcup_{n=1}^{\infty} O_n'$. (See Figure 1.)

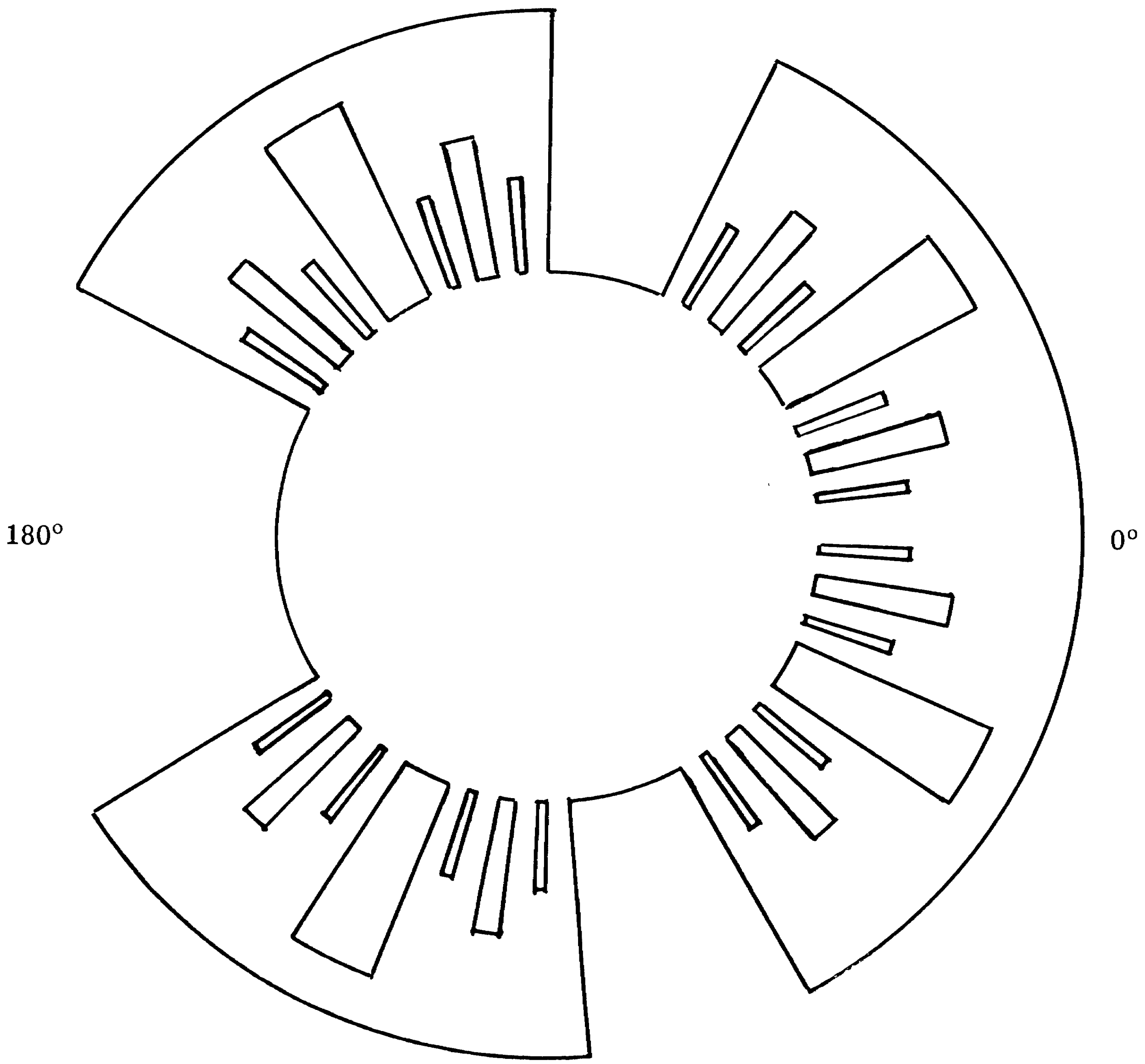

Figure 1.

The continuum E is not simply connected; we claim, however, that E has a locally connected boundary and that its visibility function is again discontinuous at the origin. The boundary is clearly locally connected at each point of the unit circle. Since $\sqrt{\rho_n} \to 0$, each boundary point of E in $\{(r, \theta): 1 < r < 2\}$ has a neighborhood intersecting exactly one of the sets O_n'. Hence, the set E has a locally connected boundary.

It is evident that the origin sees a subset of $\{(r, \theta): 1 < r \leq 2\}$ via E that has positive measure. Choose a point (r_1, θ_1) $(0 < r_1 < 1)$ in the unit disc. To establish the discontinuity of the visibility function at the origin, we again show that if (r_2, θ_2) is a point in $E \cap \{(r, \theta): 1 < r \leq 2\}$ not collinear with (r_1, θ_1) and the origin, then it is not visible from (r_1, θ_1). Without harm we may suppose that the segment joining (r_1, θ_1) to (r_2, θ_2) passes through $(1, \theta)$, where $\theta < \theta_2$. By the construction of E, we can find a strictly decreasing sequence of real numbers ϕ_n convergent to θ and such that

$$\{(r, \phi_n): 1 < r \leq 1 + \sqrt{\phi_n - \theta}\} \subset E^c$$

for all n. Clearly, these radial segments in E^c prevent (r_1, θ_1) from seeing (r_2, θ_2) via E.

The last example indicates that the visibility function may be discontinuous on the interior of a set even if the boundary of the set has measure zero.

3. A RESULT FOR OPEN SETS

THEOREM 6. *If* E *is a bounded open set in* R^2 *and* E^c *is locally connected, then* v *is continuous on* E.

Proof. We first establish that if $x \in E$, then $m(S(x)) = m(\text{cl } S(x))$. If $q \in \text{cl } S(x)$ and $p \in \text{bd } S(x) \cap \text{intv } qx$, then $pq \subset \text{bd } S(x)$. Hence, points in bd S(x) appear as closed segments on a line through x or as endpoints of maximal segments in cl S(x) with one endpoint x. The latter set has measure zero, as we have seen before.

We note that any such segment of boundary points of S(x) has the property that the point on the segment nearest x is a point of bd E. Suppose that the number of such segments of boundary points is uncountable. We can then find a sequence of these segments $\{a_n b_n\}$, where $a_n \in \text{bd } E$ for each n, convergent to a nondegenerate segment ab. Clearly, $ab \subset \text{bd } S(x)$, and a belongs to bd E. We claim that a is not a point of local connectivity of E^c.

Let $\delta = (1/2) \min \{\|a - x\|, \|b - a\|\}$. Suppose N is a connected neighborhood of a relative to E^c and $N \subset B_\delta(a)$. Clearly, we can find segments $a_{n_1} b_{n_1}$ and $a_{n_2} b_{n_2}$ in our original sequence such that $a_{n_j} \in N$ $(j = 1, 2)$ and $a_{n_1} b_{n_1}$ is between $a_{n_2} b_{n_2}$ and ab. Since b_{n_1} is a boundary point of S(x), there exists a line segment in S(x) that separates either a_1 and a_2 in N or a_1 and a in N, a contradiction. Thus the number of segments of boundary points of S(x) collinear with x is countable, so that $m(S(x)) = m(\text{cl } S(x)) = m(\text{cl } S(x) \cap E)$.

We now show that $E \cap \text{cl } S(x) \supset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} S(x_k)$, for each sequence $\{x_k\}$ in E convergent to x. If not, there exist a point $p \in E$ and a number $\varepsilon > 0$ such that $B_\varepsilon(p) \cap S(x) = \emptyset$, but $p \in S(x_k)$ for infinitely many k. The line L(x, p) determines two open half-spaces H^+ and H^-. We distinguish two cases for a point $q \in E^c \cap xp$:

(1) every neighborhood containing q contains points in $E^c \setminus xp$;

(2) there exists a neighborhood of q containing no points of $E^c \setminus xp$.

Suppose some q in $E^c \cap xp$ satisfies condition (1). If we pick our neighborhood N of q small and locally connected relative to E^c, then all points of $N \cap (E^c \setminus xp)$ lie in one open half-space, say H^+, or else $\{px_k\}$ cannot be contained in E frequently. It is now easy to see that corresponding to every point $q' \in xp \cap E^c$, there exists a neighborhood of q' containing no points of $E^c \cap H^-$. Obviously, such a neighborhood exists for each point of $E \cap xp$. The compactness of xp implies the existence of an open rectangle R in $H^- \cap E$ with base xp. Since $B_\varepsilon(p) \cap R \subset S(x)$, we conclude that $p \in \text{cl } S(x)$, a contradiction. Hence, $E \cap \text{cl } S(x) \supset \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty S(x_k)$, so that

$$m(S(x)) = m(\text{cl } S(x)) \geq m\left(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty S(x_k)\right),$$

and the upper-semicontinuity of v follows. ■

Unbounded open sets on which the visibility function is discontinuous are easy to construct. However, it is an open question whether the visibility function for a bounded open set must be continuous.

REFERENCES

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