CATEGORIES OF m-BOUNDED HAUSDORFF SPACES

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In this paper we show that the m-bounded Hausdorff spaces and the m-bounded T_3-spaces form epireflective subcategories of the category $\mathscr{H}$ of Hausdorff spaces with continuous functions. We introduce the concepts of m-normality and strong m-normality, and we show that these are sufficient (but not necessary) conditions to insure that the universal maps associated with the above epireflections be embeddings. Moreover, for a strongly m-normal space we realize the universal object in the category of m-bounded T_3-spaces as a subspace of the Wallman compactification.

Throughout the paper, all spaces will be assumed to be T_1-spaces, m will denote an infinite cardinal, and W(X) will denote the Wallman compactification of X. If F is a closed subset of X, then F^* will denote the set

$$\{u: u \text{ is a closed ultrafilter on } X \text{ and } F \in u\}$$

(the notation is that of [5, p. 167]). The collection

$$\{F^*: F \text{ is a closed subset of } X\}$$

is a base for the closed sets in W(X). The cardinality of a set A will be denoted by $|A|$, and the closure of A in X by $cl_X A$. By $S_m(X)$ we shall denote the subspace

$$\{y \in W(X): y \in cl_{W(X)} A \text{ for some } A \subseteq X \text{ with } |A| \leq m\}$$

of W(X). As in [3], a topological space X will be called *m-bounded* if the closure in X of every subset A of X with $|A| \leq m$ is compact. All m-bounded spaces are clearly countably compact. An example of a countably compact space that is not m-bounded for any infinite cardinal m is given in [2, Section 9.15].

LEMMA 1. *$S_m(X)$ is m-bounded.*

Proof. Let A be a subset of $S_m(X)$ with cardinality less than or equal to m. Since $A \subseteq S_m(X)$, each $y \in A$ is such that $y \in cl_{W(X)} A_y$ for some $A_y \subseteq X$ with $|A_y| \leq m$. Therefore

$$A \subseteq \bigcup_{y \in A} (cl_{W(X)} A_y) \subseteq cl_{W(X)}\Big(\bigcup_{y \in A} A_y\Big),$$

and this last set is a subset of $S_m(X)$, since $\left|\bigcup_{y\in A} A_y\right| \leq m$. Hence $cl_{S_m(X)} A$ is compact (being a closed subset of W(X)).

Similarly, it is easy to show that $X = S_m(X)$ if and only if X is m-bounded, and hence that $S_m(S_m(X)) = S_m(X)$.

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A space X will be called m-*normal* if disjoint closed subsets of an m-separable closed subset of X have disjoint open neighborhoods in X. (A set is said to be m-*separable* if it has a dense subset of cardinality less than or equal to m.) Clearly, an m-separable, m-normal space is normal. The Tychonoff plank (see [2, Section 8.20]) is an example of an $\aleph_0$-normal space that is not normal.

The proof of the following lemma is quite easy, and we omit it.

LEMMA 2. *An* m-*bounded Hausdorff space is* m-*normal.*

THEOREM 1. $S_m(X)$ *is a Hausdorff space if and only if* X *is* m-*normal.*

Proof. Suppose X is m-normal. Let u and v be distinct elements of $S_m(X)$. Then there exist subsets A_u and A_v of X with $|A_u| \leq m$ and $|A_v| \leq m$ such that

$$u \in cl_{W(X)} A_u \quad \text{and} \quad v \in cl_{W(X)} A_v .$$

Since u and v are distinct closed ultrafilters on X, there exist disjoint closed subsets B_u and B_v of X such that $B_u \in u$ and $B_v \in v$. Clearly,

$$B_u \cap cl_X A_u \in u \quad \text{and} \quad B_v \cap cl_X A_v \in v .$$

Furthermore, $B_u \cap cl_X A_u$ and $B_v \cap cl_X A_v$ are disjoint closed subsets of the m-separable set $cl_X(A_u \cup A_v)$; since X is m-normal, they have disjoint open neighborhoods U and V in X. Consequently, $(X - U)^*$ and $(X - V)^*$ are closed subsets of W(X) whose union is W(X) and such that $u \notin (X - U)^*$ and $v \notin (X - V)^*$. Hence $S_m(X) - (X - U)^*$ and $S_m(X) - (X - V)^*$ are disjoint open neighborhoods of u and v, respectively.

Conversely, suppose A and B are disjoint closed subsets of an m-separable closed subset F of X. Since $cl_{W(X)} F$ is contained in $S_m(X)$, it follows that $cl_{W(X)} A$ and $cl_{W(X)} B$ are compact subsets of $S_m(X)$. Furthermore, since A and B are disjoint, their closures in W(X) are also disjoint (see [5, p. 168]). If $S_m(X)$ is a Hausdorff space, $cl_{W(X)} A$ and $cl_{W(X)} B$ have disjoint open neighborhoods in $S_m(X)$; this implies that A and B have disjoint open neighborhoods in X.

A space X will be called *strongly* m-*normal* if disjoint closed subsets of X, one of which is contained in an m-separable subset of X, have disjoint open neighborhoods in X. Clearly, all strongly m-normal spaces are m-normal and regular. The Tychonoff plank is an example of a completely regular space that is $\aleph_0$-normal, but neither $\aleph_1$-normal nor strongly $\aleph_0$-normal. An example of a non-normal, strongly $\aleph_0$-normal space is given in [2, Problem 8L].

LEMMA 3. *An* m-*bounded* T_3-*space is strongly* m-*normal.*

THEOREM 2. $S_m(X)$ *is a* T_3-*space if and only if* X *is strongly* m-*normal.*

The proofs of these results are similar to the proofs of Lemma 2 and Theorem 1, and we omit them.

In the following theorem, $\mathscr{A}_m$ (respectively, $\mathscr{B}_m$) denotes the category of m-bounded Hausdorff spaces (respectively, m-bounded T_3-spaces).

THEOREM 3. $\mathscr{A}_m$ *and* $\mathscr{B}_m$ *are epireflective subcategories of the category* $\mathscr{H}$ *of Hausdorff spaces. Furthermore, for* m-*normal spaces the universal maps associated with* $\mathscr{A}_m$ *are embeddings, and for a strongly* m-*normal space* X *the embedding of* X *in* $S_m(X)$ *is the universal map associated with* $\mathscr{B}_m$.

Proof. It is clear that $\mathscr{A}_m$ and $\mathscr{B}_m$ are full subcategories of $\mathscr{H}$. In [3] it is shown that a product of m-bounded spaces is m-bounded, and it is clear that a closed subspace of an m-bounded space is m-bounded. That $\mathscr{A}_m$ and $\mathscr{B}_m$ are epireflective subcategories of $\mathscr{H}$ now follows from [4, Theorem 1.2.1].

Suppose now that X is an m-normal space. Lemma 1 and Theorem 1 imply that $S_m(X)$ is in $\mathscr{A}_m$. Since X can be embedded in an object of $\mathscr{A}_m$, it is clear that the universal map for X associated with $\mathscr{A}_m$ is an embedding.

Suppose that $f: X \to Y$ is a continuous function into an m-bounded T_3-space Y. We show that f has a continuous extension to $S_m(X)$. For each $u \in S_m(X)$, the collection

$$\mathscr{F}_u = \{cl_Y(f[A]): A \in u\}$$

has the finite-intersection property, and hence it is contained in some closed ultrafilter on Y. Since $u \in S_m(X)$, we know that $u \in cl_{W(X)} B$ for some $B \subseteq X$ with $|B| \leq m$. Because $|f[B]| \leq m$ and Y is m-bounded, $cl_Y(f[B])$ is compact. Since $u \in cl_{W(X)} B$, it follows that $cl_X B \in u$; therefore, because $cl_X B \subseteq f^{-1}[cl_Y(f[B])]$, each closed ultrafilter containing $\mathscr{F}_u$ contains a compact set, and hence converges to some point of Y. If $\mathscr{F}_u$ is contained in two distinct closed ultrafilters v and w on Y, then, since v and w must converge to distinct points y and z of Y, and since Y is a Hausdorff space, there exist disjoint open sets U and V of Y such that $y \in U$ and $z \in V$. If $f^{-1}[Y - U] \in u$, then $cl_Y(f[f^{-1}[Y - U]]) \in \mathscr{F}_u$. However, $cl_Y(f[f^{-1}[Y - U]]) \subseteq Y - U$; therefore no closed ultrafilter that contains $\mathscr{F}_u$ converges to the point y. Conversely, if $f^{-1}[Y - U] \notin u$, there exists $C \in u$ such that $C \cap f^{-1}[Y - U] = \emptyset$; that is to say, $C \subseteq f^{-1}[U]$, and therefore $f^{-1}[Y - V] \in u$. However, this implies that no ultrafilter containing $\mathscr{F}_u$ converges to z. Hence there is a unique closed ultrafilter v_u in Y that contains $\mathscr{F}_u$. Since Y is a Hausdorff space, there is a unique element $y_u \in Y$ to which v_u converges.

Define the mapping $\hat{f}: S_m(X) \to Y$ by $\hat{f}(u) = y_u$. Clearly, the restriction of $\hat{f}$ to X is f. Let A be a closed subset of Y. We shall show that $\hat{f}^{-1}[A]$ is closed. Suppose that $u \in S_m(X) - \hat{f}^{-1}[A]$. Then, since Y is regular, there exist disjoint open subsets U_u and V_u of Y such that $A \subseteq U_u$ and $\hat{f}(u) \in V_u$. Furthermore, there exists $B \in u$ such that $B \subseteq f^{-1}[V_u]$; for otherwise, $f^{-1}[Y - V_u] \in u$, and hence $\hat{f}(u) \in Y - V_u$. Thus $u \notin (X - f^{-1}[V_u])^*$. Conversely, if $v \notin (X - f^{-1}[V_u])^*$, then there exists $D \subseteq f^{-1}[V_u]$ such that $D \in v$, and hence $\hat{f}(v) \notin A$. Thus

$$\hat{f}^{-1}[A] = S_m(X) \cap \left(\bigcap \{(X - f^{-1}[V_u])^*: \hat{f}(u) \notin A\} \right);$$

clearly, this is a closed subset of $S_m(X)$.

It follows from Lemma 1 and Theorem 2 that if X is strongly m-normal, then $S_m(X)$ is in $\mathscr{B}_m$ and is therefore the universal object for X associated with $\mathscr{B}_m$.

Remarks. 1. We note that in our proof, the existence and continuity of the extension $\hat{f}$ depend only on X being a T_1-space. It is easy to show that a continuous Hausdorff image of an m-bounded space is m-bounded. It follows immediately that if X is a Hausdorff space, the universal object for X in $\mathscr{B}_m$ is the universal object in the category of T_0-spaces for the "regularization" (see [6]) of $S_m(X)$.

2. Results for completely regular spaces, analogous to those above, appear in [7]. Thus it is worth noting that not all strongly m-normal T_1-spaces are completely regular. An example of a strongly $\aleph_0$-normal space that is not completely

regular can be constructed along the lines of Example 2.4.4 of [1], the only change being the replacement of the space Z of the example by the space Ω of problem 8L of [2].

3. Since each $T_{3\frac{1}{2}}$-space X can be embedded in the m-bounded T_3-space βX, the universal maps for X associated with $\mathscr{A}_m$ and $\mathscr{B}_m$ must be embeddings. It would be interesting to find necessary and sufficient conditions for these universal maps to be embeddings.

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